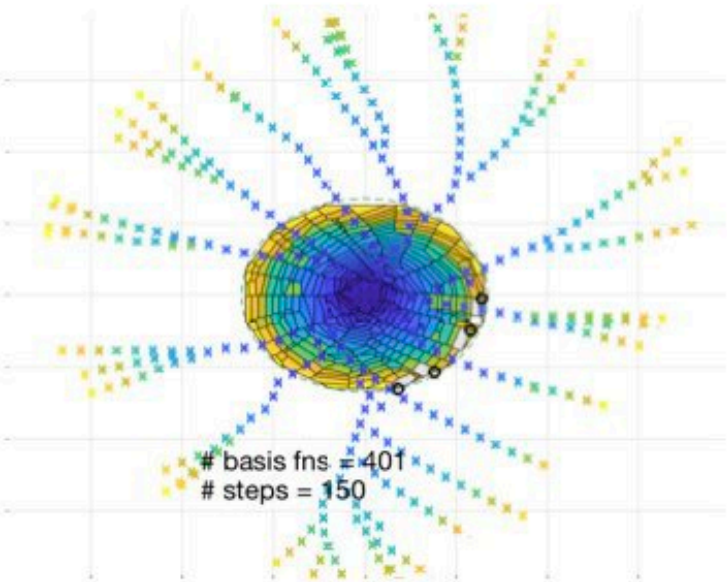
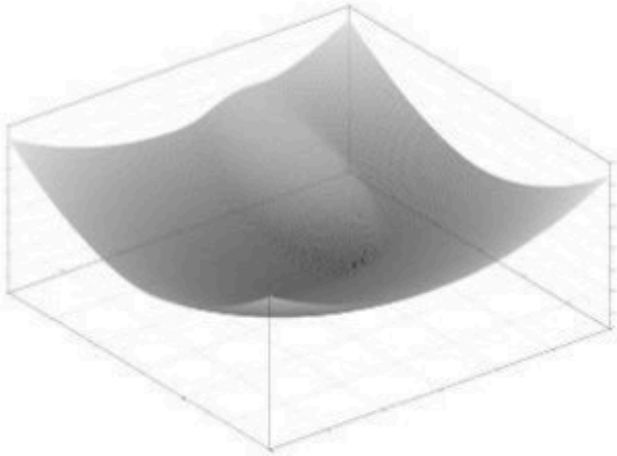
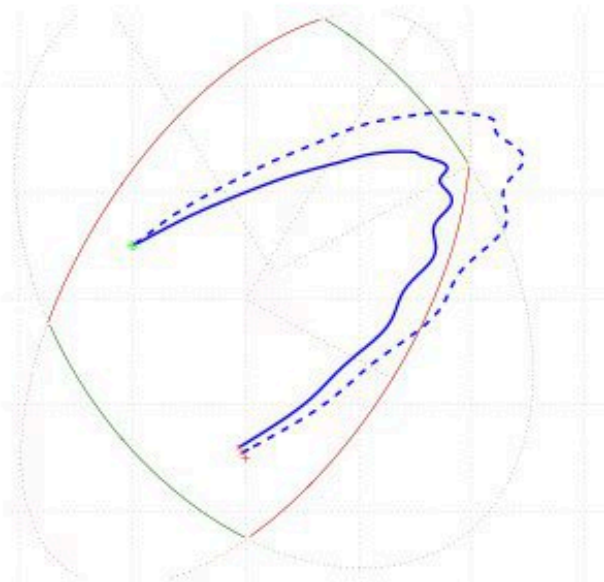
Sparse approximation methods for optimal control

incl. basis adaptation for max-plus eigenvector methods







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Outline

"Sparse approximation methods for optimal control"

- Optimal control problems.
  - Hamilton-Jacobi-Bellman PDEs.
  - Feedback solutions.
  - Curse-of-dimensionality.

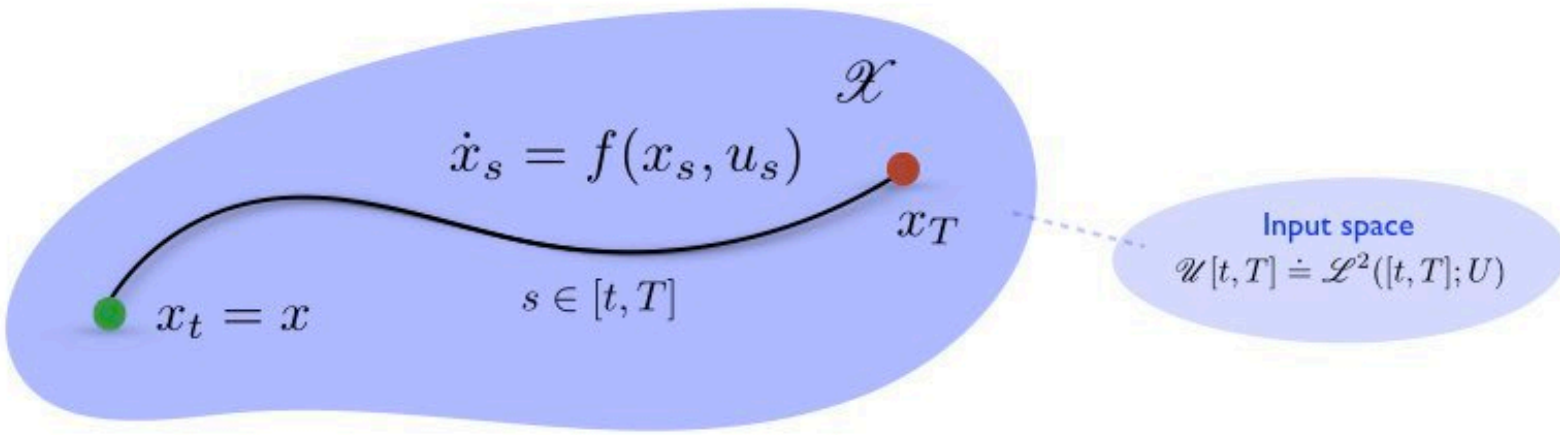
- Towards computational tractability.
  - Max-plus methods.
  - Optimization methods.
  - Relaxation methods.

Optimal control problem

Cost or payoff $$J_t(x,u) \doteq J_t[\Psi](x,u) \doteq \int_t^T L(x_s,u_s)\,ds + \Psi(x_T)$$

subject to $$x_s = x + \int_t^s f(x_\sigma,u_\sigma)\,d\sigma \in \mathscr{X} \qquad \text{(dynamics)}$$

$$f(x,u) \doteq f(x) + g(x)\,u$$



Value $$W_t(x) \doteq \inf_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u) \qquad L(x,u) \doteq l(x) + \tfrac{1}{2}\langle u, \mathcal{M}\,u\rangle \qquad \text{(cost)}$$

$$W_t(x) \doteq \sup_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u) \qquad L(x,u) \doteq l(x) - \tfrac{1}{2}\langle u, \mathcal{M}\,u\rangle \qquad \text{(payoff)}$$

Optimal control problem

Cost or payoff $$J_t(x,u) \doteq J_t[\Psi](x,u) \doteq \int_t^T L(x_s,u_s)\,ds + \Psi(x_T)$$

Value $$W_t(x) \doteq (\mathcal{S}_{T-t}\Psi)(x) \doteq \inf_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u)$$

dynamic programming $$W_t = \mathcal{S}_\tau\, W_{t+\tau}$$

semigroup $$\mathcal{S}_{\tau+\sigma} = \mathcal{S}_\tau\, \mathcal{S}_\sigma$$

backward DP

$$\mathcal{S}_0 = \mathcal{I}$$

unique viscosity solution

M.G. Crandall, P.-L. Lions
N.N. Subbotin

Hamiltonian $$H(x,p) \doteq -\inf_{u\in U}\{L(x,u) + \langle p,\, f(x,u)\rangle\}$$
$$= -l(x) - \langle p,\, f(x)\rangle + \tfrac{1}{2}\langle p,\, g(x)\,\mathcal{M}^{-1}\,g(x)\,p\rangle$$

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x,\nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

feedback solution

$$K_t(x) \doteq \arg\min_{u\in U}\{L(x,u) + \langle \nabla_x W_t(x),\, f(x,u)\rangle\}$$

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Optimal control problem

Cost or payoff $$J_t(x,w) \doteq J_t[\Psi](x,w) \doteq \int_0^t L(x_s,w_s)\,ds + \Psi(x_t)$$

Value $$W_t(x) \doteq (\mathcal{S}_t\Psi)(x) \doteq \sup_{w\in\mathscr{W}[0,t]} J_t[\Psi](x,w)$$

dynamic programming $$W_{t+\tau} = \mathcal{S}_\tau\, W_t$$

semigroup $$\mathcal{S}_{t+\tau} = \mathcal{S}_\tau\, \mathcal{S}_t$$

forward DP

$$\mathcal{S}_0 = \mathcal{I}$$

unique viscosity solution

M.G. Crandall, P.-L. Lions
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Hamiltonian $$H(x,p) \doteq -\sup_{w\in W}\{L(x,w) + \langle p,\, f(x,w)\rangle\}$$
$$= -l(x) - \langle p,\, f(x)\rangle - \tfrac{1}{2\gamma^2}\,\langle p,\, g(x)\,g(x)'\,p\rangle$$

HJB PDE $$0 = \frac{\partial W_t}{\partial t}(x) + H(x,\nabla_x W_t(x)) \qquad W_0(x) = \Psi(x)$$

feedback solution

$$K_s(x) \doteq \arg\max_{w\in W}\{L(x,w) + \langle \nabla_x W_{t-s}(x),\, f(x,w)\rangle\}$$

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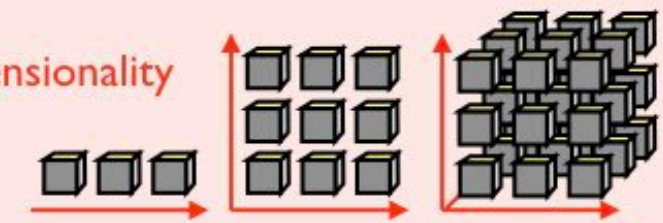
Optimal control problem

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

HJB PDE $$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_0(x) = \Psi(x)$$

solution method (grid-based)

curse-of-dimensionality



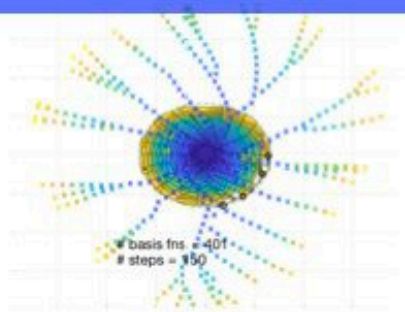
dimension 3 or 4 at best ..... intractability

Optimal control problem

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

HJB PDE $$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_0(x) = \Psi(x)$$

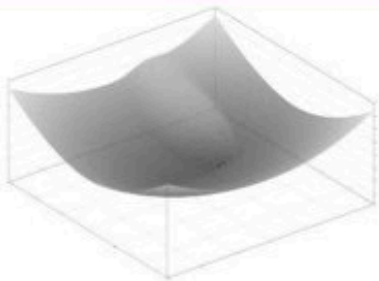
max-plus methods



sparse finite elements

[D., McEneaney]

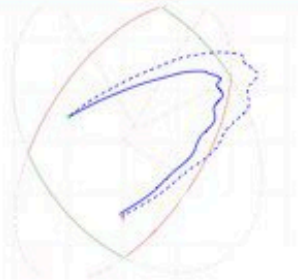
optimization methods



optimization over characteristics

[D., Yegorov, Grune]

relaxation methods



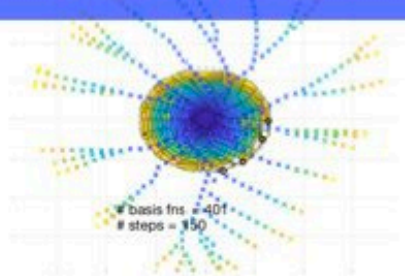
decomposition into LQR problems

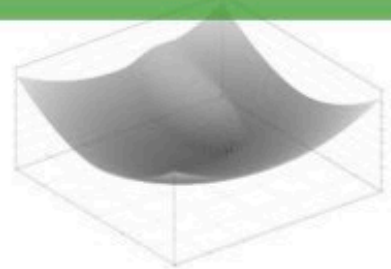
[D., Basco, McEneaney]

Optimal control problem

HJB PDE	$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x))$	$W_T(x) = \Psi(x)$
HJB PDE	$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x))$	$W_0(x) = \Psi(x)$

dynamic programming	characteristic flow	semiconvex relaxation
max-plus linearity & semiconvexity	minimax solutions	game & DREs
max-plus methods	optimization methods	relaxation methods





Optimal control problem

HJB PDE $$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_0(x) = \Psi(x)$$

dynamic programming	characteristic flow	semiconvex relaxation
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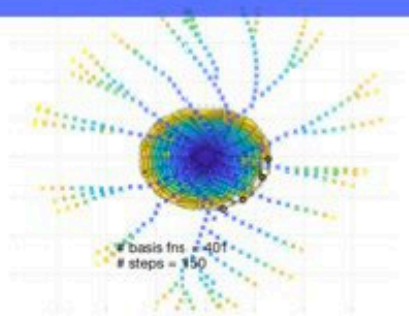
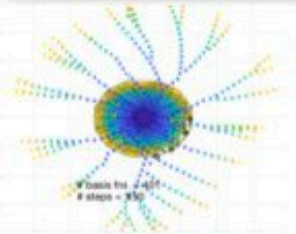


Max-plus methods



Value $$W_t(x) \doteq (\mathcal{S}_t \Psi)(x) \doteq \sup_{w \in \mathscr{W}[0,t]} J_t[\Psi](x,w)$$

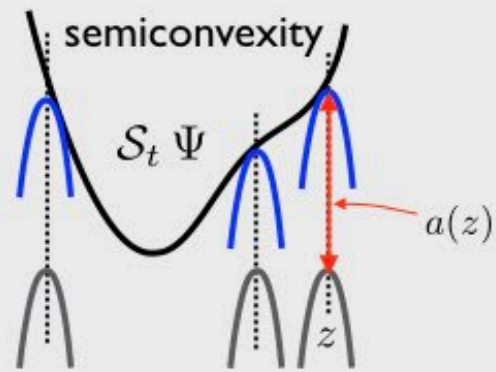
semigroup

$$\mathcal{S}_{t+\tau} = \mathcal{S}_t \, \mathcal{S}_\tau$$

max-plus linearity

$$\mathcal{S}_t(\psi \oplus c \otimes \phi) = \mathcal{S}_t \, \psi \oplus [c \otimes \mathcal{S}_t \, \phi]$$

max-plus $a \oplus b \doteq \max(a,b)$ $\quad a \otimes b \doteq a + b$



quadratic basis $\varphi(\cdot, z)$

$$a \doteq \mathcal{D}_\varphi W_t \doteq -\sup_{x \in \mathscr{X}} \{\varphi(x,\cdot) - W_t(x)\}$$

$$W_t = \mathcal{D}_\varphi^{-1} a = \sup_{x \in \mathscr{X}} \{\varphi(\cdot, z) + a(z)\}$$

max-plus fundamental solution semigroups

$$\{\mathcal{G}_\tau^\oplus\}_{\tau \geq 0}, \quad \{\mathcal{B}_\tau^\oplus\}_{\tau \geq 0}$$

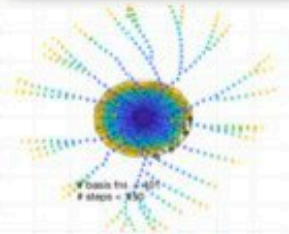
[McEneaney],
[D. et. al.]

e.g. [D.] "An adaptive max-plus eigenvector method for continuous time optimal control problems." In Numerical Methods for Optimal Control Problems, Ed. Falcone et al., Springer, 2018.

value propagation $$W_{t+\tau} = \mathcal{G}_\tau^\oplus \, W_t \qquad \text{(primal)}$$

$$W_{t+\tau} = \mathcal{D}_\varphi^{-1} \, \mathcal{B}_\tau^\oplus \, \mathcal{D}_\varphi \, W_t \qquad \text{(dual)}$$

Max-plus methods



value propagation $W_{k\tau} = \mathcal{D}_\varphi^{-1} \mathcal{B}_\tau^{\oplus} \mathcal{D}_\varphi W_{(k-1)\tau}$

(e.g. dual)

$$W_{k\tau} = \mathcal{D}_\varphi^{-1} a_k$$

dual space *function* iteration

$$a_k \doteq \mathcal{B}_\tau^{\oplus} a_{k-1} \qquad \mathcal{B}_\tau^{\oplus} a \doteq \sup_{\zeta \in \mathscr{X}} \{B_\tau(\cdot, \zeta) + a(\zeta)\}$$

with $B_\tau(z, \zeta) \doteq (\mathcal{D}_\varphi \mathcal{S}_\tau \varphi(\cdot, \zeta))(z)$

[McEneaney], [D. et. al.]

quadratic basis for (uniformly) semiconvex functions

$$\{\varphi(\cdot, z_i)\}_{i \in \mathbb{N}} \qquad \{z_i\}_{i \in \mathbb{N}} \text{ dense in } \mathscr{X}$$

coordinate representation $[e_k]_i \doteq a_k(z_i)$

dual space *coordinate* iteration $e_k = B_\tau \otimes e_{k-1}$

(max-plus)

(abuse of notation) $[B_\tau]_{ij} \doteq B_\tau(z_i, z_j)$

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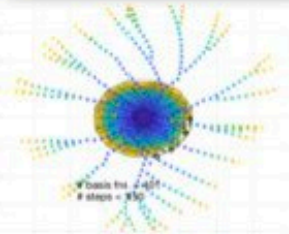
Max-plus methods



value propagation $W_{k\tau} = \mathcal{D}_\varphi^{-1} \mathcal{B}_\tau^{\oplus} \mathcal{D}_\varphi W_{(k-1)\tau}$

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[McEneaney], [D. et. al.]

$$[e_k]_i \doteq a_k(z_i) \qquad \{z_i\}_{i \in \mathbb{N}} \text{ dense in } \mathscr{X}$$

$$\begin{aligned}
[e_k]_i &= [\mathcal{B}_\tau^{\oplus} a_{k-1}](z_i) = \sup_{\zeta \in \mathscr{X}} \{B_\tau(z_i, \zeta) + a_{k-1}(\zeta)\} \\
&= \max_{j \in \mathbb{N}} \{B_\tau(z_i, z_j) + a_{k-1}(z_j)\} \\
&= \max_{j \in \mathbb{N}} \{B_\tau(z_i, z_j) + [e_{k-1}]_j\}
\end{aligned}$$

dual space *coordinate* iteration $e_k = B_\tau \otimes e_{k-1}$

(max-plus)

(abuse of notation) $[B_\tau]_{ij} \doteq B_\tau(z_i, z_j)$

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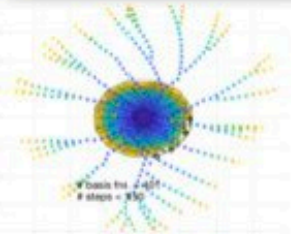
Max-plus methods



dual space *coordinate* iteration $e_k = B_\tau \otimes e_{k-1}$

$e_k \in \overline{\mathbb{R}}^\infty \quad [B_\tau]_{ij} \doteq B_\tau(z_i, z_j)$

truncate basis

i.e. $\widehat{e}_k \in \overline{\mathbb{R}}^\nu \quad \{z_i\}_{i \le \nu} \quad \nu \in \mathbb{N}$

max-plus method

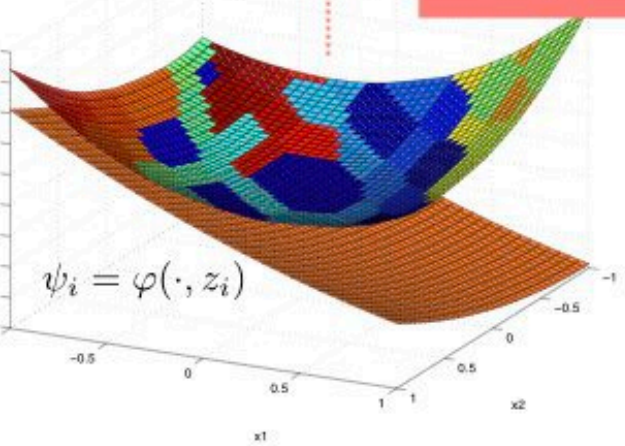
$$W_{k\tau}(x) \approx \widehat{W}_{k\tau}(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_k]_i \qquad \tau \in (0, \tau^*)$$

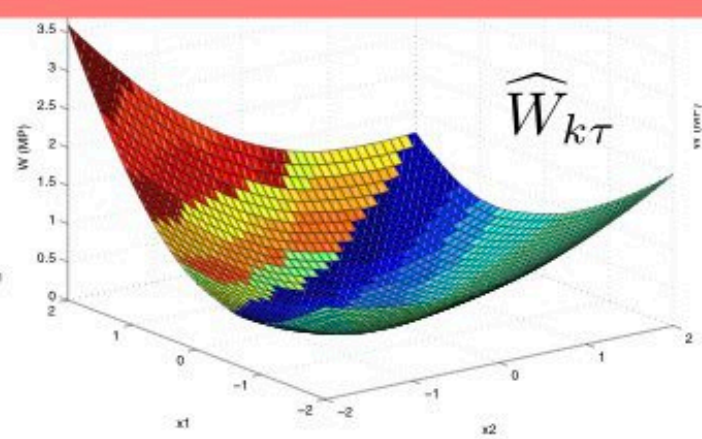
$$[\widehat{e}_0]_i = [\mathcal{D}_\varphi \Psi](z_i)$$

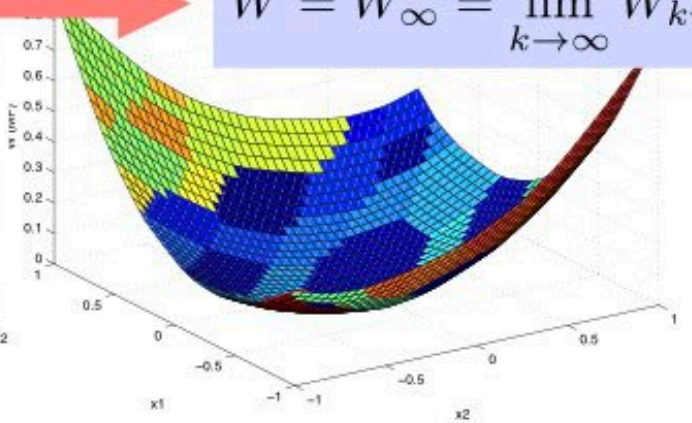
with $\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1} \qquad k \in \mathbb{N}$

iterate

$$\widehat{W} \doteq \widehat{W}_\infty \doteq \lim_{k \to \infty} \widehat{W}_{k\tau}$$







(with fixed basis and known $\widehat{B}_\tau \in \mathbb{R}^{\nu \times \nu}$)

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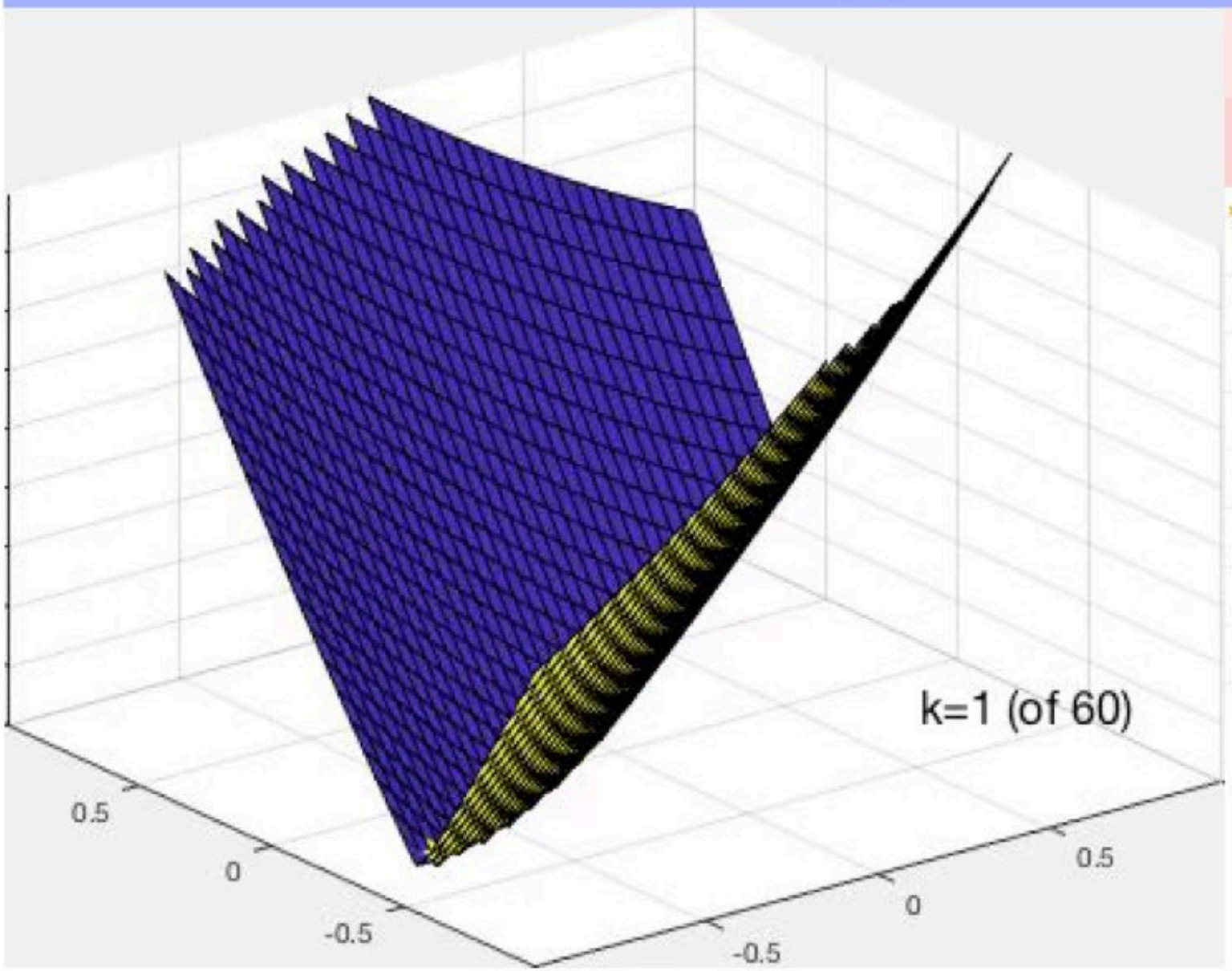
Max-plus methods

max-plus method $\widehat{W}(x) \doteq \widehat{W}_\infty(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_\infty]_i$

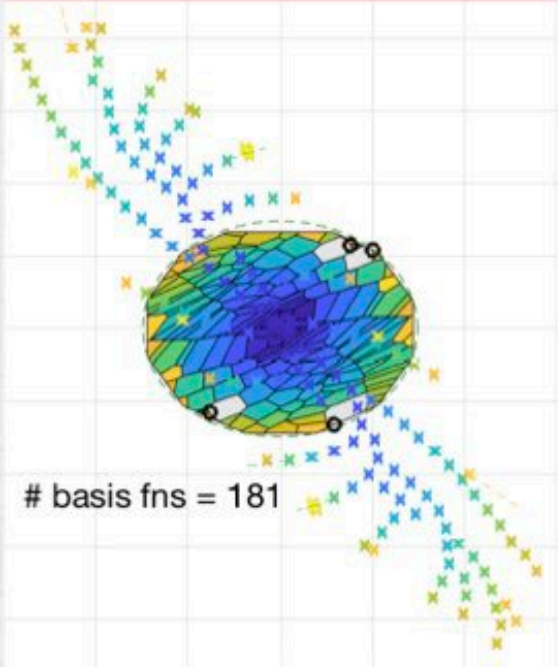
$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

$\widehat{e}_\infty \doteq \lim_{k\to\infty} \widehat{e}_k$



with fixed basis

and known $\widehat{B}_\tau \in \mathbb{R}^{\nu\times\nu}$



Max-plus methods

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$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

$\widehat{e}_\infty \doteq \lim_{k\to\infty} \widehat{e}_k$

computation of $\widehat{B}_\tau \in \mathbb{R}^{\nu\times\nu}$?

$\{z_i\}_{i\leq\nu} \quad \nu \in \mathbb{N}$

$[\widehat{B}_\tau]_{ij} = (\mathcal{D}_\varphi \, \mathcal{S}_\tau \, \varphi(\cdot, z_j))(z_i)$

$\varphi(x, z) \doteq \frac{1}{2}\langle x - z, \mathcal{M}(x - z)\rangle \quad \mathcal{M} = \mathcal{M}' \in \mathbb{R}^{n\times n}$

Max-plus methods

max-plus method $\widehat{W}(x) \doteq \widehat{W}_\infty(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_\infty]_i$

$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

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e.g. $L(x, w) \doteq l(x) - \frac{\gamma^2}{2}|w|^2 \quad g(x) = \sigma \in \mathbb{R}^{n\times m} \quad H(x, p) = -l(x) - \langle p, f(x)\rangle - \frac{1}{2\gamma^2}\langle p, \sigma\,\sigma'\, p\rangle$

$V_t(x) \doteq (\mathcal{S}_t\, \varphi(\cdot, z_j))(x)$ (auxiliary optimal control problem)

HJB PDE $0 = \frac{\partial V_t^j}{\partial t}(x) + H(x, \nabla_x V_t^j(x)) \quad V_0^j(x) = \varphi(x, z_j)$

$t \in [0, \tau]$

$x \in \mathbb{R}^n$

$j \leq \nu$

Max-plus methods

max-plus method $\widehat{W}(x) \doteq \widehat{W}_\infty(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_\infty]_i$

$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

$\widehat{e}_\infty \doteq \lim_{k\to\infty} \widehat{e}_k$

computation of $\widehat{B}_\tau \in \mathbb{R}^{\nu\times\nu}$?

$\{z_i\}_{i\leq\nu} \quad \nu \in \mathbb{N}$

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e.g. $L(x, w) \doteq l(x) - \frac{\gamma^2}{2}|w|^2 \quad g(x) = \sigma \in \mathbb{R}^{n\times m} \quad H(x, p) = -l(x) - \langle p, f(x)\rangle - \frac{1}{2\gamma^2}\langle p, \sigma\,\sigma'\, p\rangle$

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$t \in [0, \tau]$

$x \in \mathbb{R}^n$

$j \leq \nu$

series solution — $\tau > 0$ small

$$[\widehat{B}_\tau]_{ij} = (\mathcal{D}_\varphi\, V_\tau^j)(z_i) = \inf_{x\in\mathbb{R}^n}\{V_\tau^j(x) - \varphi(x, z_i)\} = V_\tau^j(X_\tau^j) - \varphi(X_\tau^j, z_i)$$

parameter selection + symbolic computations + nonlinear ODE for $s \mapsto X_s^j$ via RK45

(curse-of-dimensionality)

$O(\nu^2)$ nonlinear ODE solutions via RK45

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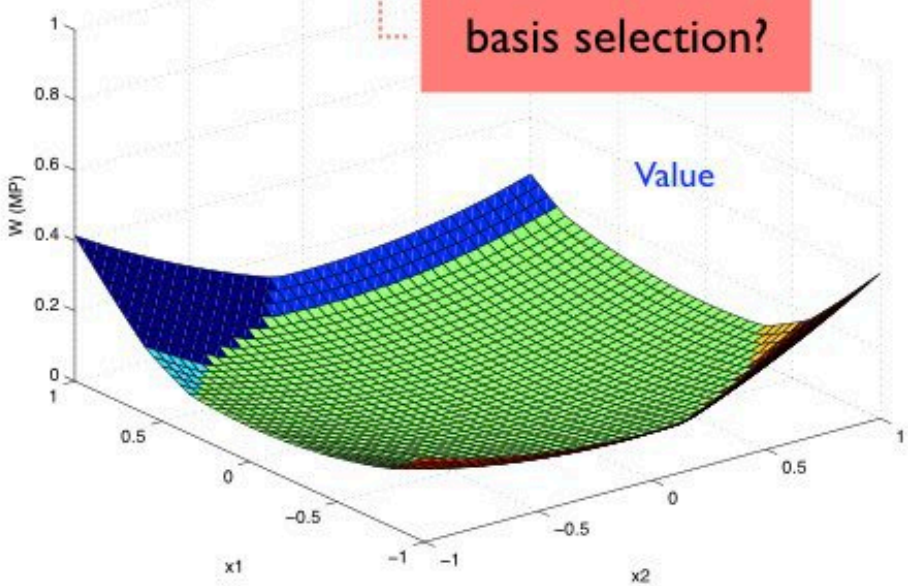
Basis adaptation

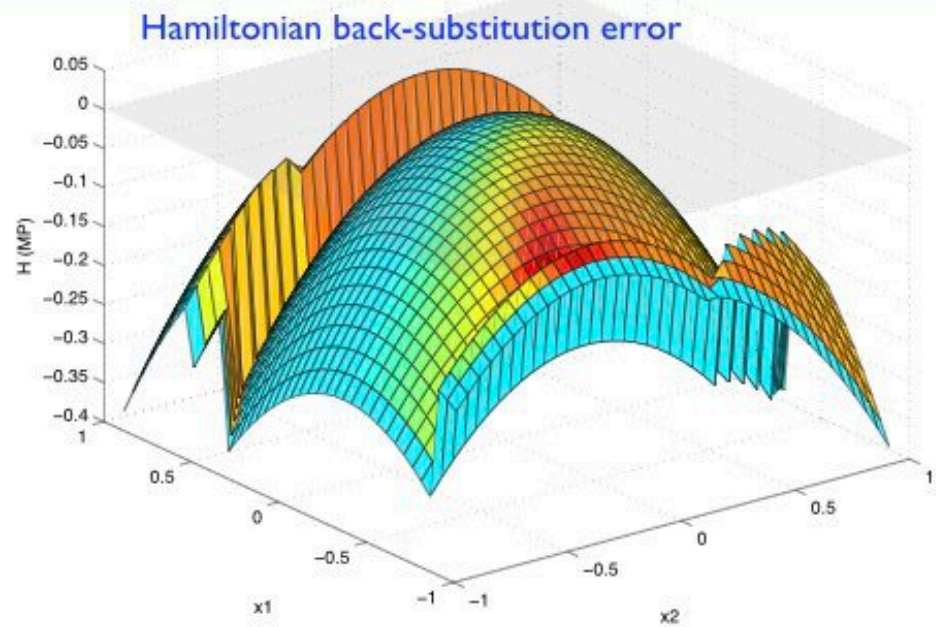
max-plus method $\widehat{W}(x) \doteq \widehat{W}_\infty(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_\infty]_i$

$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

$\widehat{e}_\infty \doteq \lim_{k\to\infty} \widehat{e}_k$

basis selection?





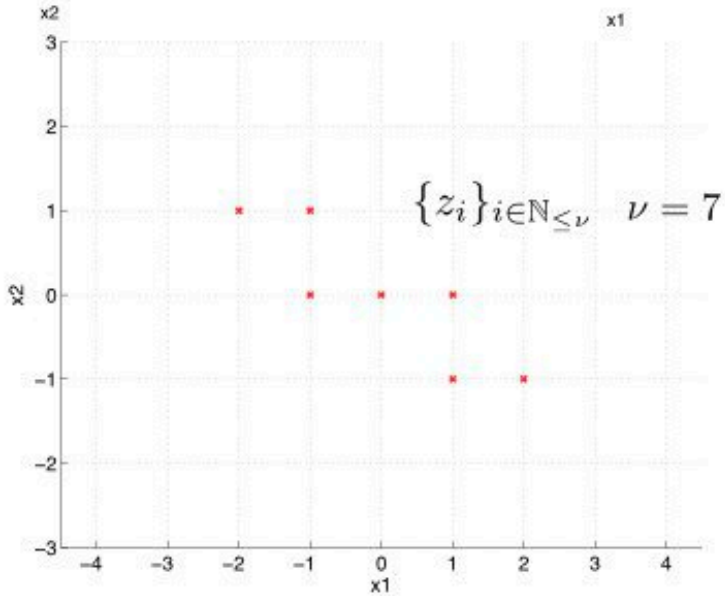
Example 1

linear dynamics

$$A \doteq \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \quad B \doteq \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

$$l(\xi) \doteq \tfrac{1}{2}|\xi|^2 \qquad \gamma \doteq 2$$

$$\mathcal{M} \doteq 0.2\,\mathcal{I}$$



$\{z_i\}_{i\in\mathbb{N}_{\leq\nu}} \quad \nu = 7$

7 basis functions

$$\psi_i(x) \doteq \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

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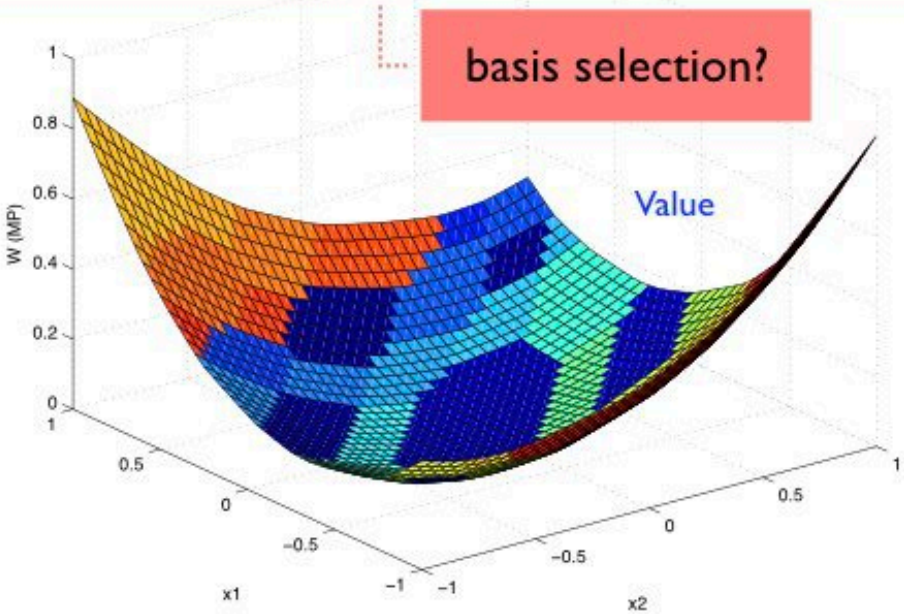
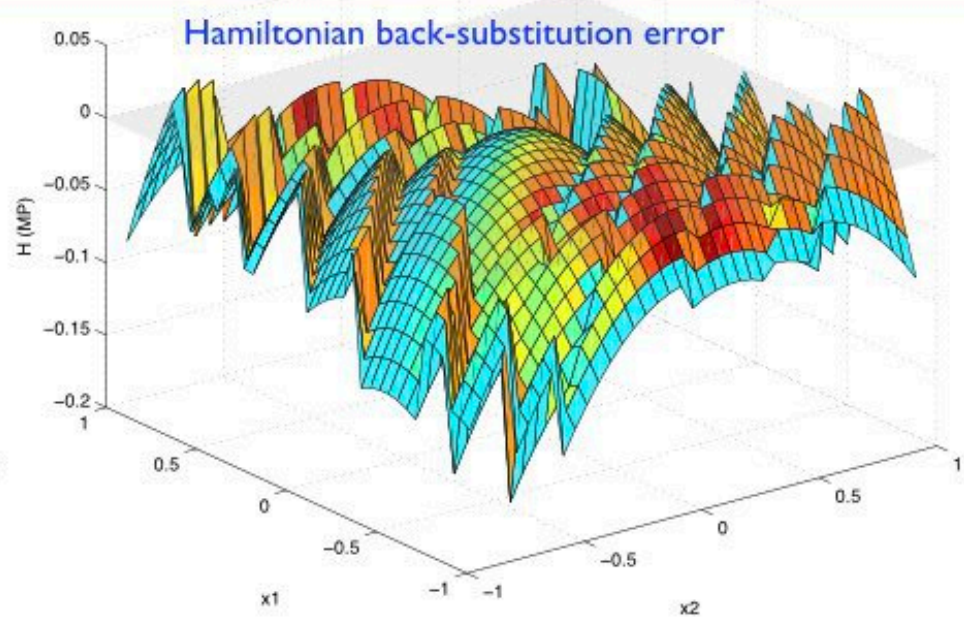
Basis adaptation

max-plus method $\widehat{W}(x) \doteq \widehat{W}_\infty(x) = \bigoplus_{i=1}^{\nu} \varphi(x, z_i) \otimes [\widehat{e}_\infty]_i$

$\widehat{e}_k = \widehat{B}_\tau \otimes \widehat{e}_{k-1}$

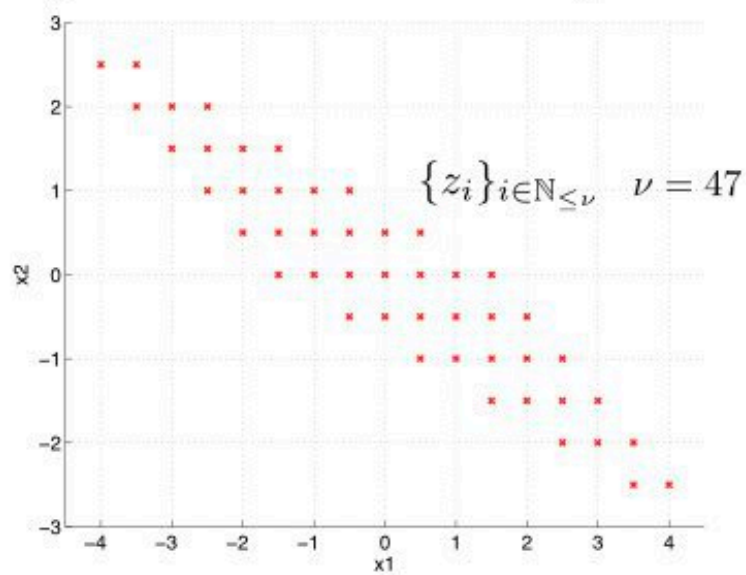
$\widehat{e}_\infty \doteq \lim_{k\to\infty} \widehat{e}_k$





Example 1

linear dynamics



47 basis functions

$\psi_i(x) \doteq \varphi(x, z_i) = \frac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$

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Basis adaptation

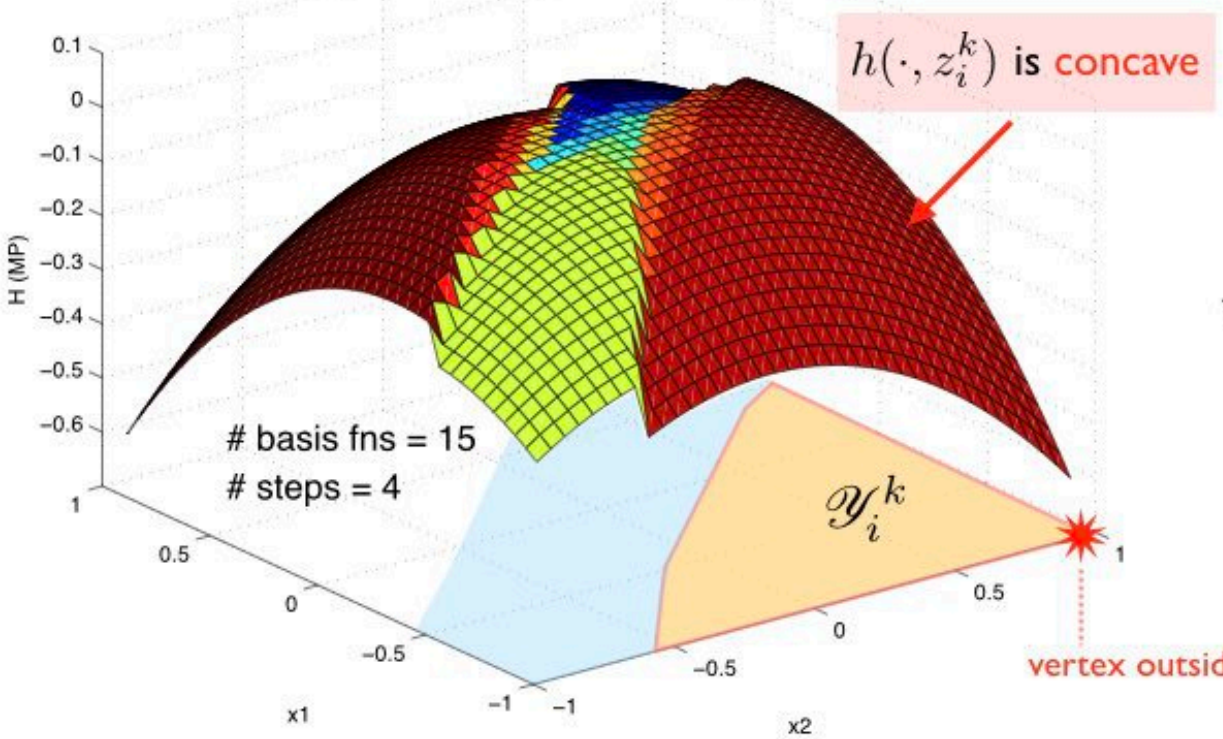
max-plus method $\widehat{W}^k(x) \doteq \widehat{W}^k_\infty(x) = \bigoplus_{i=1}^{\nu^k} \varphi(x, z^k_i) \otimes [\widehat{e}^k_\infty]_i$ (using k^{th} iterate of basis)

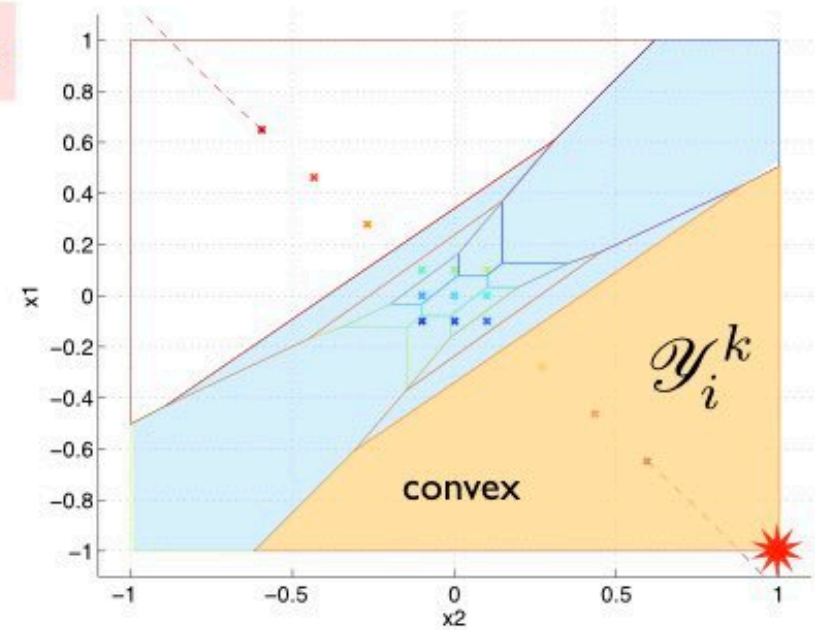
local approximation $\widehat{W}^k(x) = \psi^k_i(x) + [\widehat{e}^k_\infty]_i$ $\quad x \in \mathscr{Y}^k_i \quad i \in \mathbb{N}_{\leq \nu^k}$

gradient $\nabla_x \widehat{W}^k(x) = \nabla_x \psi^k_i(x) = \mathcal{M}\,(x - z^k_i)$

Hamiltonian back-substitution error $h(x, z^k_i) \doteq H(x, \nabla_x \psi^k_i(x))$

Hamiltonian $H(x,p) = -l(x) - \langle p,\, f(x)\rangle - \frac{1}{2\gamma^2}\,\langle p,\, \sigma\,\sigma'\,p\rangle$





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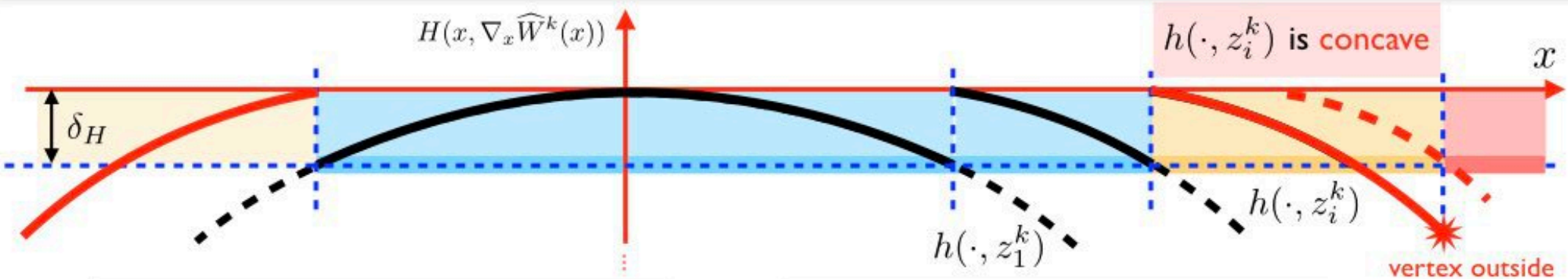
Basis adaptation



Gradient descent

$$\dot{z}_\eta = h_y(z_\eta) \qquad z_0 = z_i \qquad \epsilon \in \mathbb{R}_{>0}$$

$$h_y(z) \doteq -\operatorname{sgn} h(y,z) \frac{\nabla_z h(y,z)}{\|\nabla_z h(y,z)\| + \epsilon}$$

···· or ····

Characteristics

$$\dot{\xi}_\eta = \nabla_p H(\xi_\eta, \pi_\eta) \qquad \xi_0 = y$$

$$\dot{\pi}_\eta = -\nabla_x H(\xi_\eta, \pi_\eta) \qquad \pi_0 = \mathcal{M}(y - z_i)$$

$$z_\eta = \xi_\eta - \mathcal{M}^{-1} \pi_\eta$$

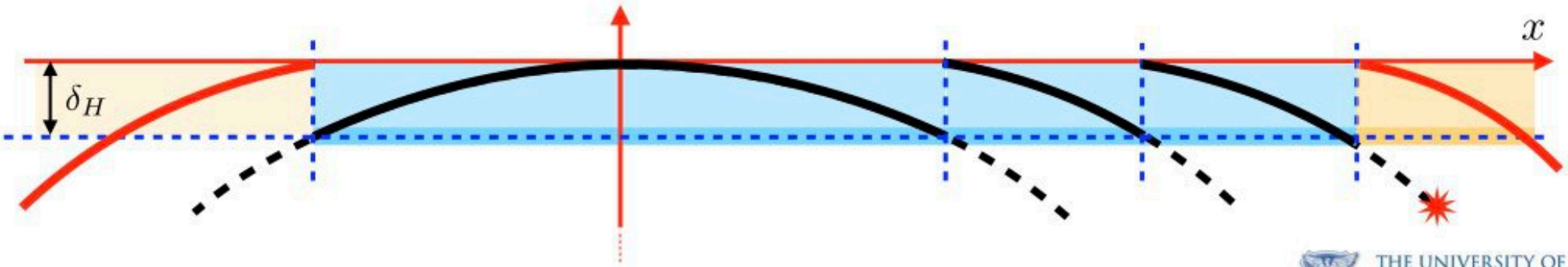
$$\eta \in \mathbb{R}_{\geq 0}$$

Add a new basis function

$$z_i^+ \doteq z_{(\eta_i^+)}$$

$$\eta_i^+ \doteq \sup \left\{ \eta \in (0, \bar{\eta}] \;\middle|\; \begin{array}{c} |H(y, p(y, z_\eta))| < (1-\mu)\,\delta_H\,, \\ |H(y, p(y, z_\eta)) - H(y, p(y, z_i))| < \mu\,\delta_H \end{array} \right\}$$

$\bar{\eta} \in \mathbb{R}_{>0}$, $\mu \in (0,1)$ a-priori fixed



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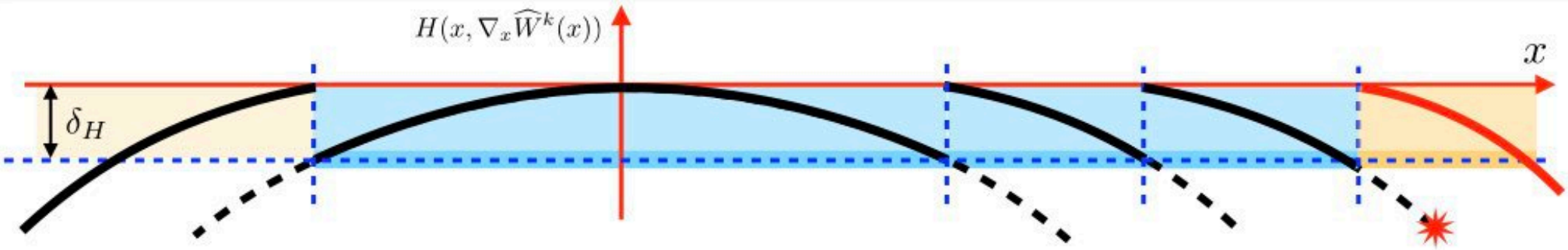
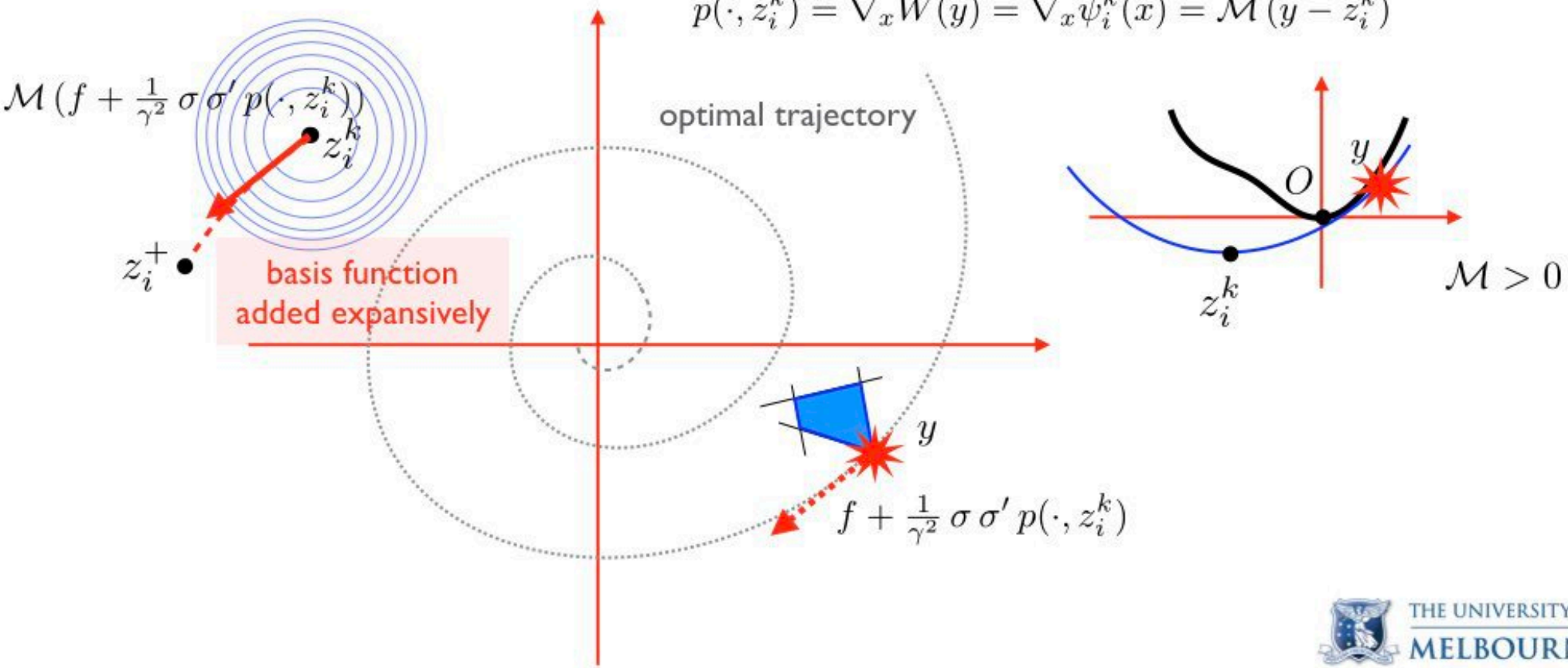
Basis adaptation



gradient descent $\nabla_z h(\cdot, z_i^k) = \mathcal{M}\left(f + \sigma\,\sigma'\,p(\cdot, z_i^k)\right)$

$$p(\cdot, z_i^k) \doteq \nabla_x \widehat{W}(y) = \nabla_x \psi_i^k(x) = \mathcal{M}\,(y - z_i^k)$$



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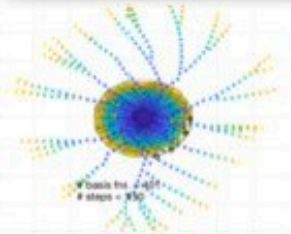
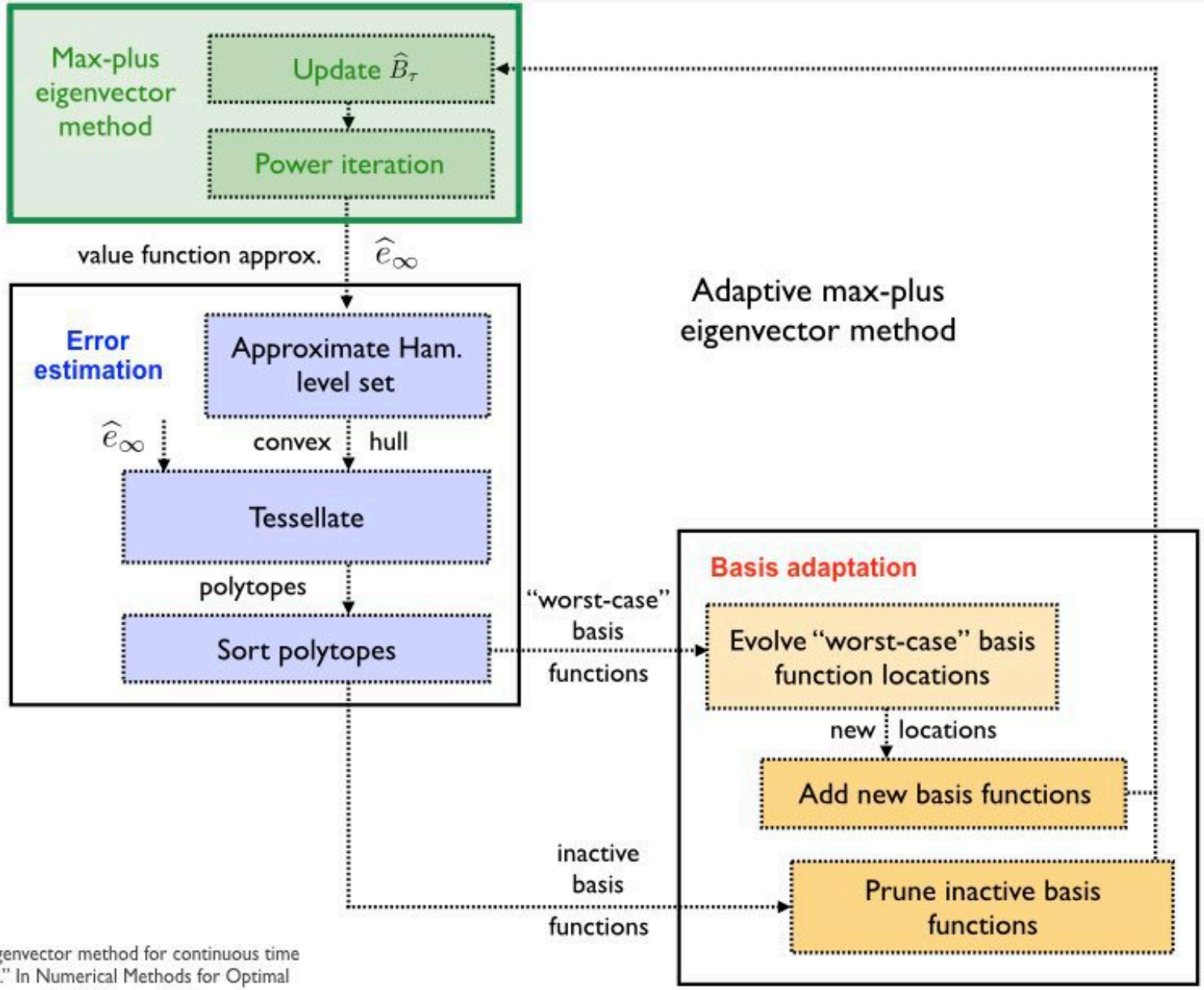
Basis adaptation





[D.] "An adaptive max-plus eigenvector method for continuous time optimal control problems." In Numerical Methods for Optimal Control Problems, Ed. Falcone et al., Springer, 2018.

[D.] "Basis adaptation for an eigenvector method arising in optimal control." Proc. MTNS (Hong Kong), 2018.

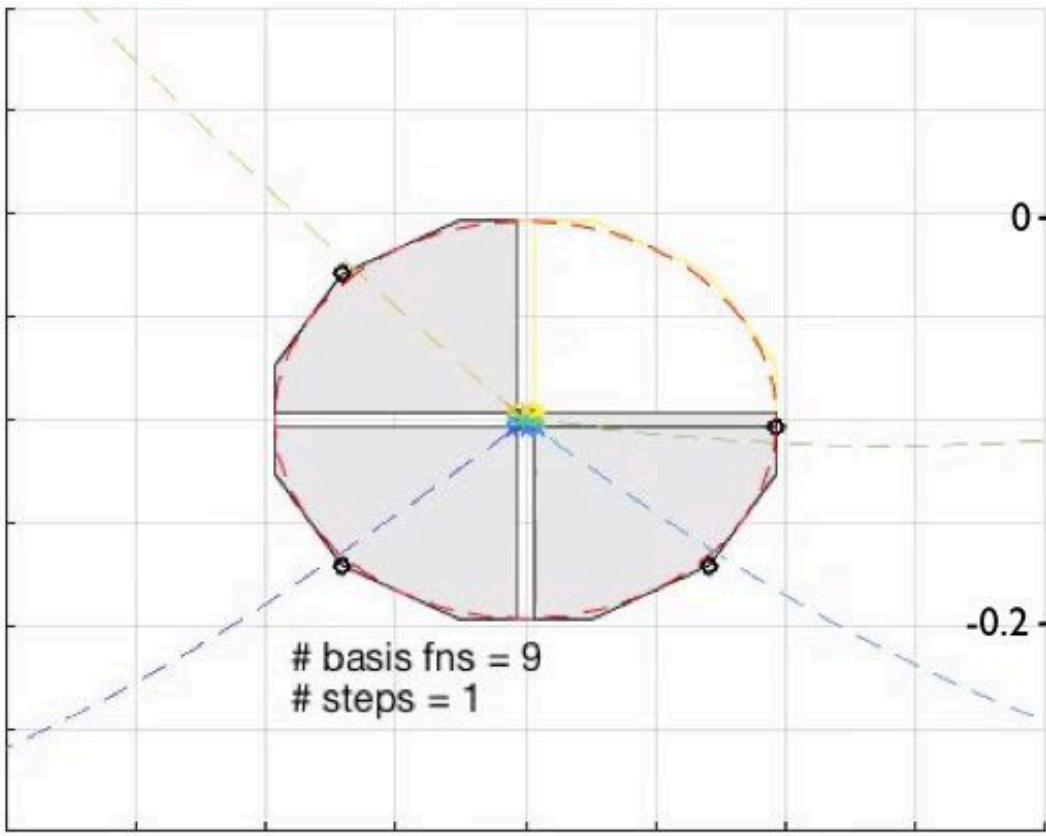
Examples

Example

nonlinear dynamics

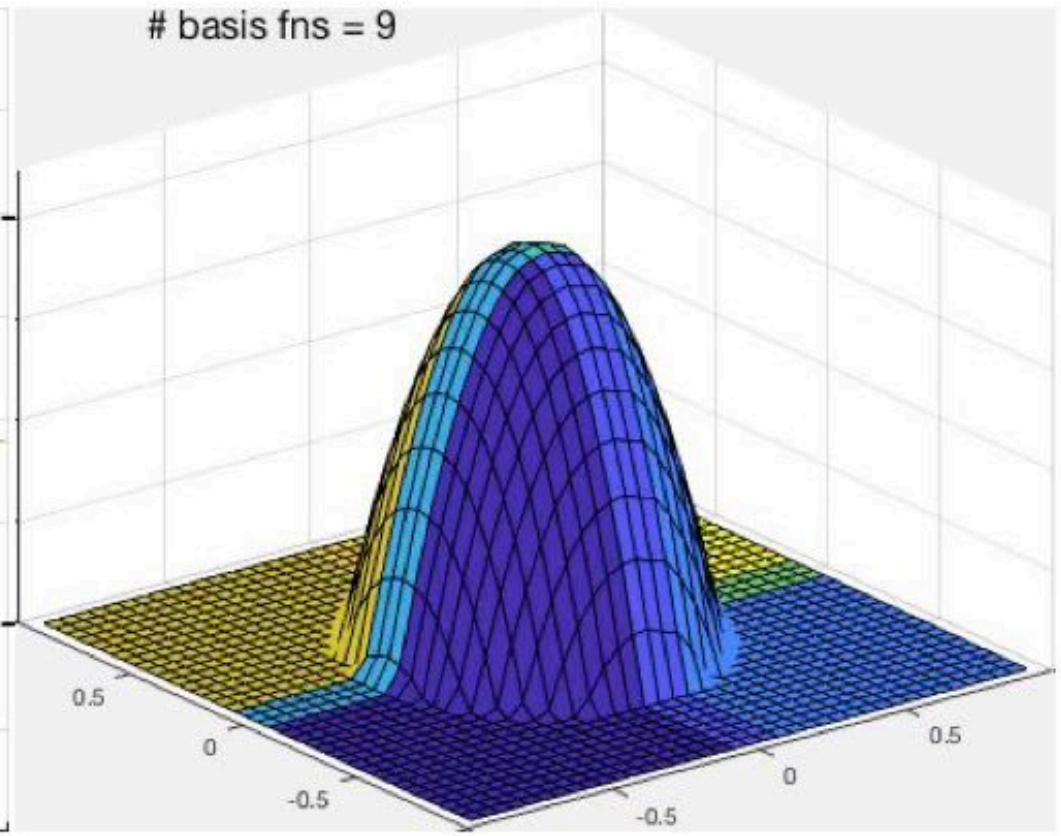
$$\dot{\xi}_s = f(\xi_s) + w_s \qquad f(\xi) \doteq \begin{pmatrix} -2\,\xi_1\,[1 + \frac{1}{2}\,\tan^{-1}(3\xi_2^2/2)] \\ \frac{1}{2}\,\xi_1 - 3\,\xi_2\,\exp(-\xi_1/3) \end{pmatrix}$$

$$l(\xi) \doteq \tfrac{1}{2}\,|\xi|^2 \qquad \gamma^2 \doteq 1 \qquad \mathcal{M} \doteq -0.1\,\mathcal{I} \qquad \boxed{\delta_H = 0.05}$$



basis & tessellation

new basis functions per step = 4



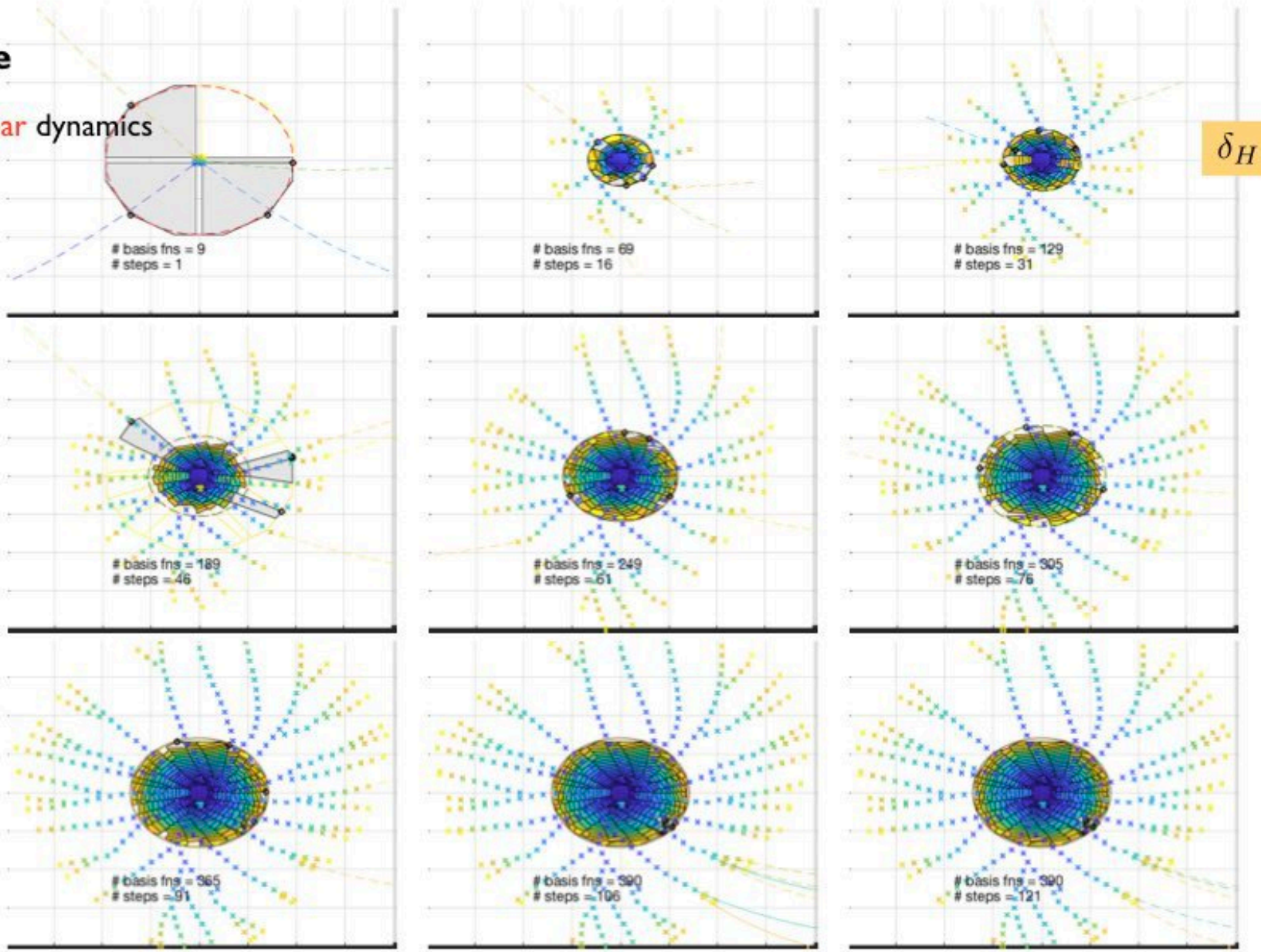
Hamiltonian

(characteristics)

Examples

Example

nonlinear dynamics



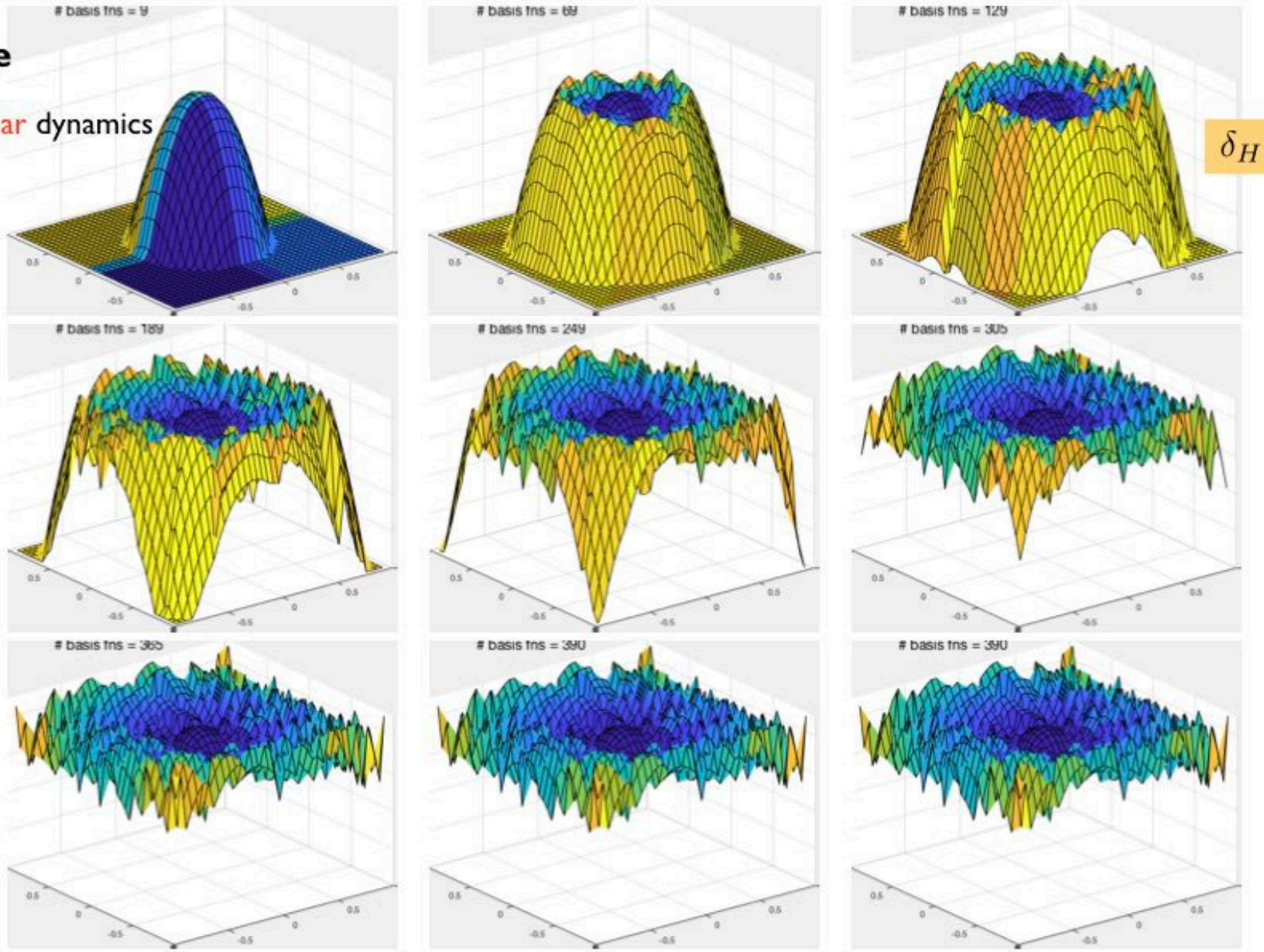
$\delta_H = 0.05$

(characteristics)

Examples

Example

nonlinear dynamics



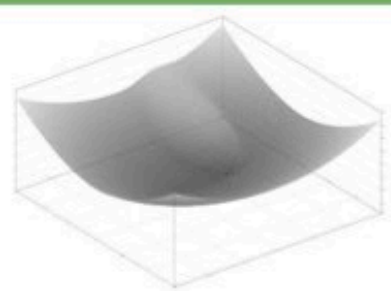
$\delta_H = 0.05$

(characteristics)

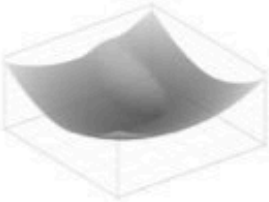
Optimal control problem

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

dynamic programming	characteristic flow	semiconvex relaxation
max-plus linearity & semiconvexity	minimax solutions	game & DREs
max-plus methods	optimization methods	relaxation methods



Optimization methods



HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

unique minimax solution of HJB PDE [Subbotin] Springer 1995

$$W_t(x) = \inf_{p_t \in \mathbb{R}^n} \left\{ \Psi(x_T) + \int_t^T \langle p_s, \nabla_p H(x_s, p_s) \rangle - H(x_s, p_s)\, ds \right\}$$

optimization over characteristics

with
$$\dot{x}_s = -\nabla_p H(x_s, p_s) \qquad x_t = x$$
$$\dot{p}_s = \nabla_x H(x_s, p_s) \qquad p_t \text{ given}$$

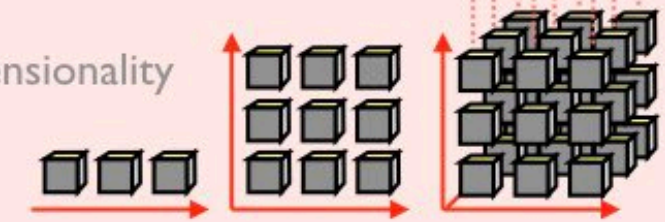
[Darbon, Osher, et al] arXiv 2017

[Yegorov, D.] AMO 2018
Theorems 2.13 & 3.8

one optimization problem per initial state

independent problems

curse-of-dimensionality



initial states
$x \in \{ x_i \}_{i \in \mathcal{I}}$

parallel computation

sampled approximation of value function

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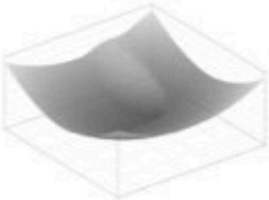
Optimization methods



Value $$W_t(x) = \sup_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

e.g. **Eikonal equation**

[Darbon, Osher, et al] arXiv 2017

[Yegorov, D.] AMO 2018

cost $$J_t(x,u) \doteq \Psi(x_T) \qquad \Psi(x) \doteq \tfrac{1}{2}\left(\langle x, A\,x\rangle_{\mathbb{R}^n} - 1\right)$$

$$A \doteq \mathrm{diag}(\tfrac{1}{4}, 1, \tfrac{1}{2}, \cdots, \tfrac{1}{2}) \in \mathbb{R}^{n\times n}$$

dynamics $$\dot{x}_s = c(x_s)\,u_s \qquad s \in [t,T] \qquad x_t = x$$

$$u_s \in U \doteq \{u \in \mathbb{R}^n \,:\, |u| \le 1\}$$

$$c(x) \doteq 1 + 3\,\exp(-4\,|x - (1,1,0,\cdots,0)|^2_{\mathbb{R}^n})$$

Hamiltonian $$H(x,p) = -\sup_{u\in U}\langle p,\, c(x)\,u\rangle = -c(x)\,|p|$$

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) - c(x)\,|\nabla_x W_t(x)| \qquad W_T(x) = \Psi(x)$$

$$(t,x) \in [0,T] \times \mathbb{R}^n$$

curse-of-dimensionality

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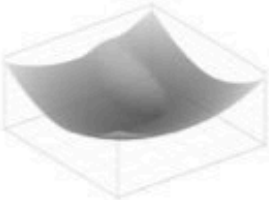
Optimization methods



Value $$W_t(x) = \sup_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

e.g. Eikonal equation

[Darbon, Osher, et al] arXiv 2017

[Yegorov, D.] AMO 2018

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) - c(x)\,|\nabla_x W_t(x)|$$

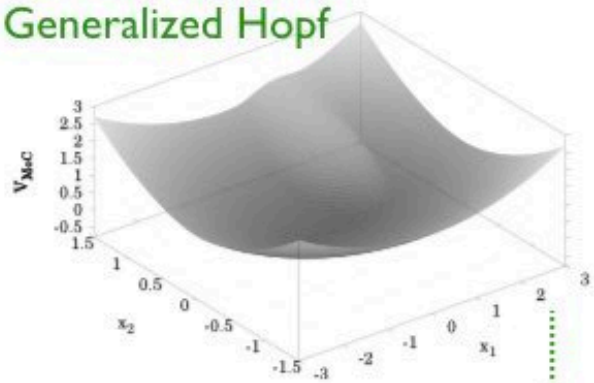
$$W_T(x) = \Psi(x)$$

$$(t,x) \in [0,T] \times \mathbb{R}^n$$

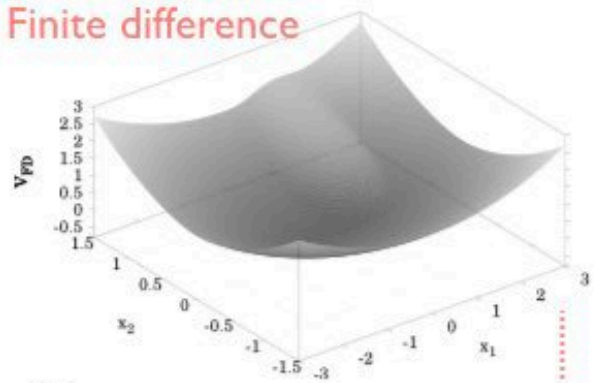
State space dimension $n \doteq 2$

$$T - t = 0.5$$

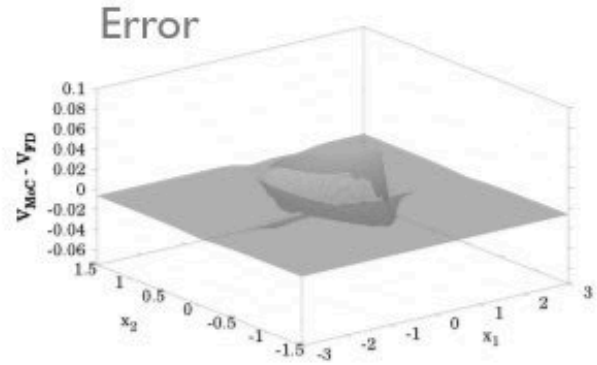
Generalized Hopf



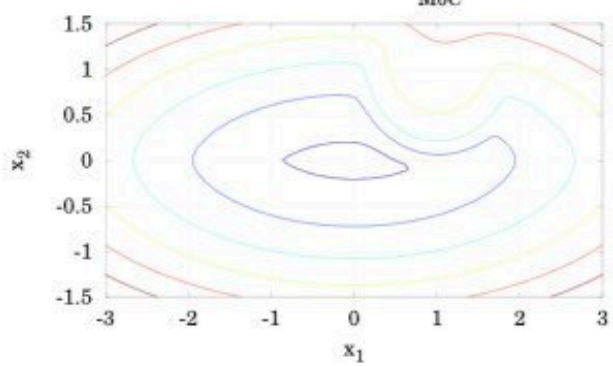
Finite difference



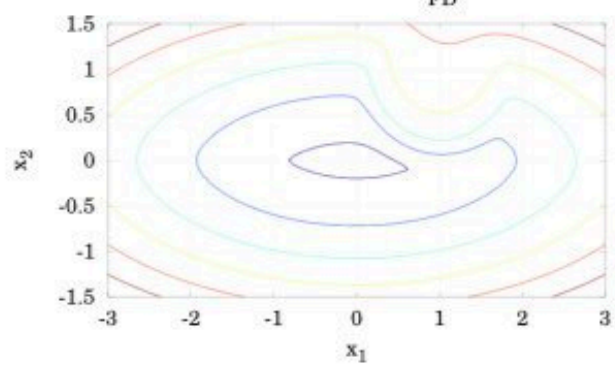
Error



Level sets of V_{MoC}



Level sets of V_{FD}



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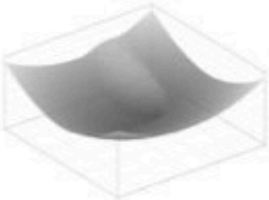
Optimization methods



Value $$W_t(x) = \sup_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

e.g. **Eikonal equation**

[Darbon, Osher, et al] arXiv 2017

[Yegorov, D.] AMO 2018

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) - c(x)\,|\nabla_x W_t(x)|$$

$$W_T(x) = \Psi(x)$$

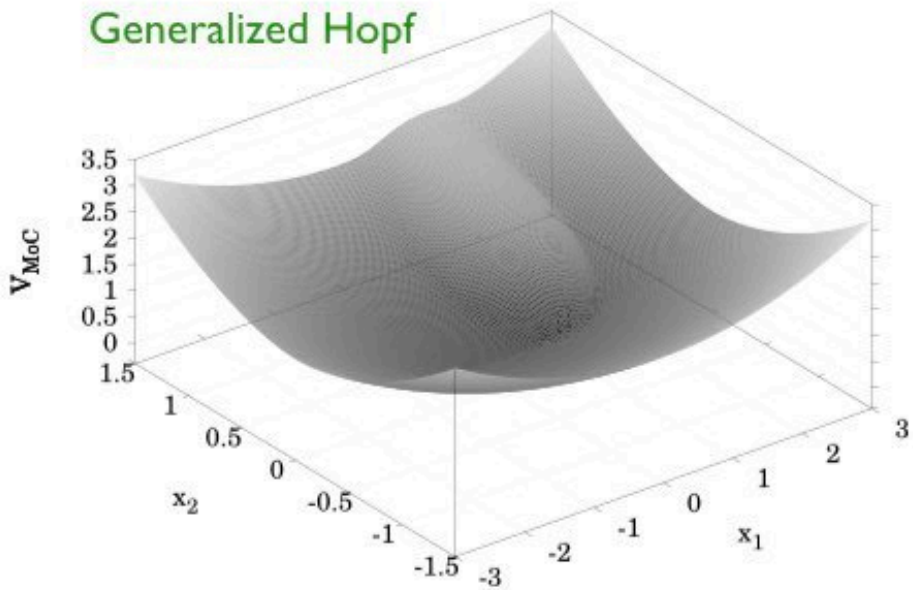
$$(t,x) \in [0,T] \times \mathbb{R}^n$$

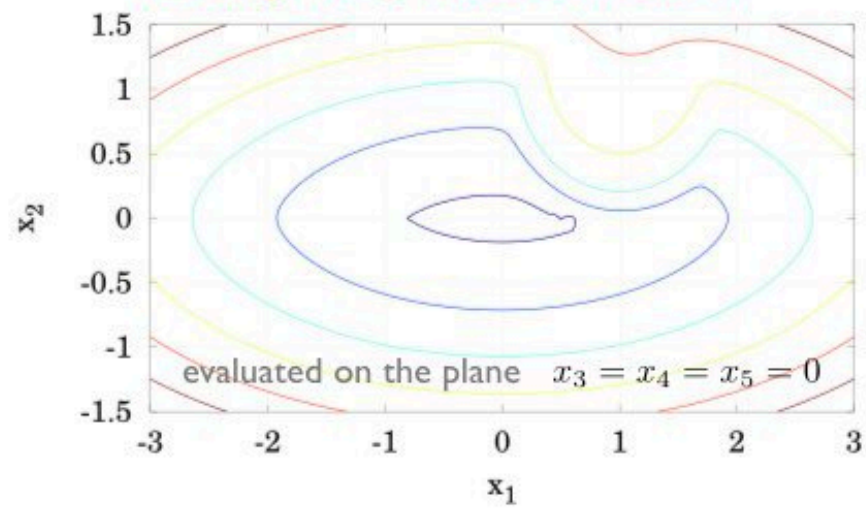
State space dimension $n \doteq 5$

$$T - t = 0.5$$

too high for grid based methods

Generalized Hopf





- [Yegorov, D.] "Perspectives on characteristics based curse-of-dimensionality-free numerical approaches to solving Hamilton Jacobi equations." AMO, 2018.
- [Yegorov, D., Grune] "Synthesis of control Lyapunov functions and stabilizing feedback strategies using exit time optimal control." arXiv:1906.02703, 2019.

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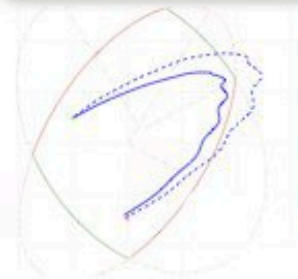
Optimal control problem

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

dynamic programming	characteristic flow	semiconvex relaxation
max-plus linearity & semiconvexity	minimax solutions	game & DREs
max-plus methods	optimization methods	relaxation methods

Relaxation methods



Value $$W_t(x) \doteq (\mathcal{S}_{T-t}\Psi)(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$$

$$J_t(x,u) \doteq \int_t^T l(x_s) + \tfrac{1}{2}\langle u_s,\, \mathcal{M}\, u_s\rangle\, ds + \Psi(x_T)$$

$$\dot{x}_s = A_s\, x_s + B_s\, u_s$$

(linear dynamics)

$$0 = -\frac{\partial W_t}{\partial t}(x) - l(x) - \langle \nabla_x W_t(x),\, A_t\, x\rangle + \tfrac{1}{2}\langle \nabla_x W_t(x),\, B_t\, \mathcal{M}^{-1}\, B_t'\, \nabla_x W_t(x)\rangle$$

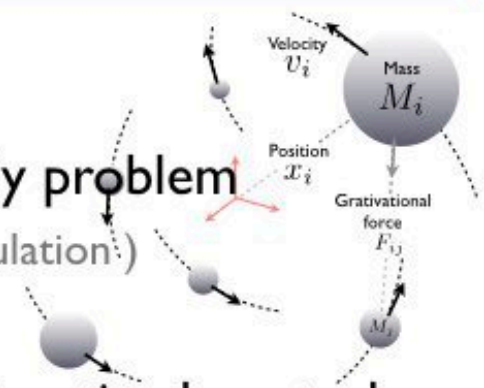
HJB PDE

$$W_T(x) = \Psi(x)$$

non-quadratic but semiconvex

e.g. gravitational N-body problem

(least action formulation)



state constrained optimal control

(etc …)

semiconvex relaxation — decomposition into LQR problems

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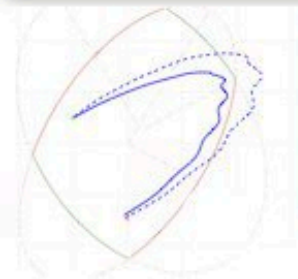
Relaxation methods



Value $$W_t(x) \doteq (\mathcal{S}_{T-t}\Psi)(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$$

$$J_t(x,u) \doteq \int_t^T l(x_s) + \tfrac{1}{2}\langle u_s, \mathcal{M}\, u_s\rangle\, ds + \Psi(x_T)$$

Illustration

$$l(x) \doteq \sum_{i=1}^N \nu_i\, \Phi \circ q_i(x)\,, \quad \Phi(\rho) \doteq \begin{cases} \phi(\rho) & \rho \in [0, b^2) \\ +\infty & \text{otherwise} \end{cases}, \quad \phi \text{ convex}, \quad \lim_{\rho \uparrow b^2} \phi(\rho) = +\infty$$

$$\sum_{i=1}^N \nu_i = 1\,, \quad q_i(x) \doteq \langle x - \bar{x}_i,\, E_i\,(x - \bar{x}_i)\rangle\,, \quad E_i \in \mathbb{R}^{n\times n}_{>0}, \quad \bar{x}_i \in \mathbb{R}^n$$

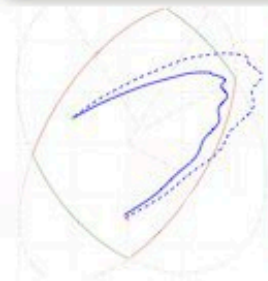
$$\Psi(x) \doteq \tfrac{1}{2}\langle x - z, P_t\,(x - z)\rangle\,, \quad P_t \in \mathbb{R}^{n\times n}_{\geq 0}, \quad z \in \mathbb{R}^n$$

$$A_t \doteq 0, \quad B_t \doteq I, \quad \mathcal{M} \doteq I$$

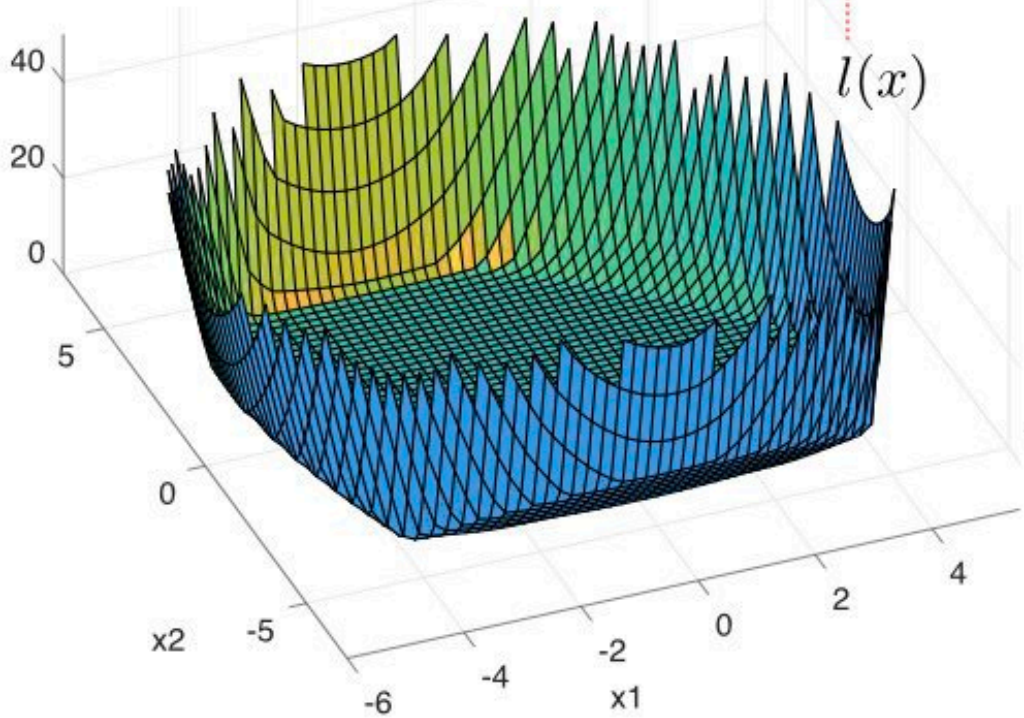
$$0 = -\frac{\partial W_t}{\partial t}(x) - \sum_{i=1}^N \nu_i\, \Phi \circ q_i(x) + \tfrac{1}{2}\,|\nabla_x W_t(x)|^2 \qquad W_T(x) = \tfrac{1}{2}\langle x - z,\, P_T\,(x - z)\rangle$$

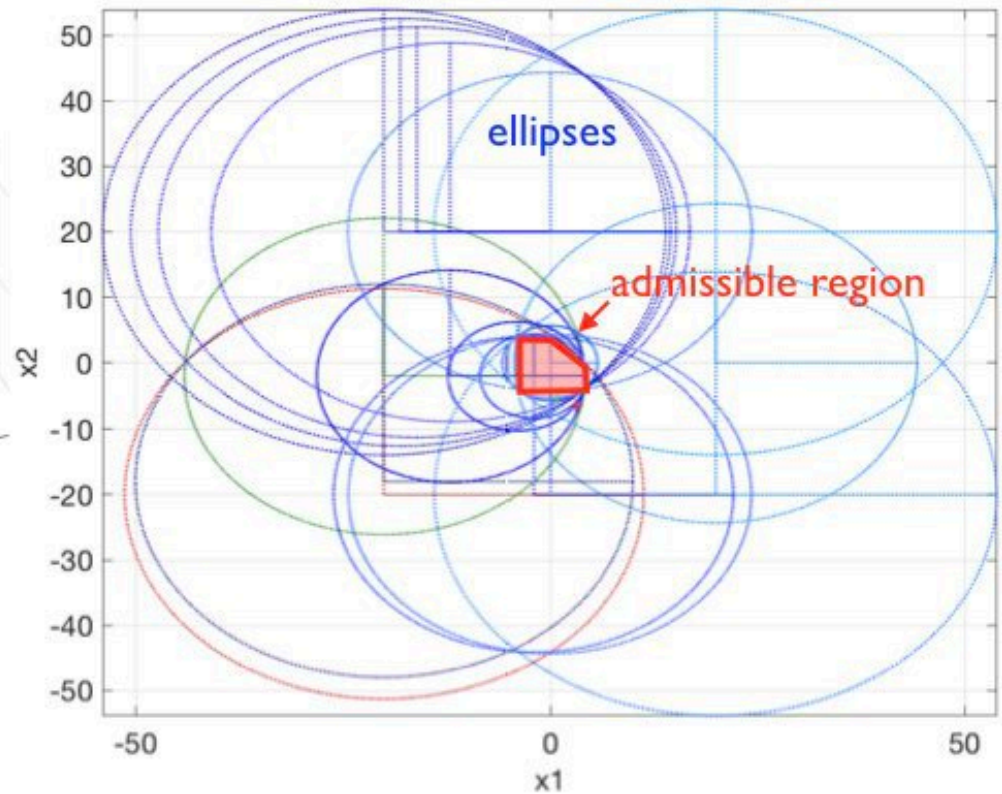
semiconvex relaxation — decomposition into LQR problems

Relaxation methods

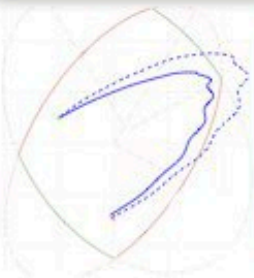


$$0 = -\frac{\partial W_t}{\partial t}(x) - \sum_{i=1}^{N} \nu_i \, \Phi \circ q_i(x) + \tfrac{1}{2} \left|\nabla_x W_t(x)\right|^2 \qquad W_T(x) = \tfrac{1}{2} \left\langle x - z,\, P_T\,(x - z)\right\rangle$$





Relaxation methods



$$0 = -\frac{\partial W_t}{\partial t}(x) - \sum_{i=1}^{N} \nu_i \, \Phi \circ q_i(x) + \tfrac{1}{2} \left|\nabla_x W_t(x)\right|^2 \qquad W_T(x) = \tfrac{1}{2} \langle x - z, \, P_T \, (x - z) \rangle$$

$$\Phi(\rho) \doteq \begin{cases} \phi(\rho) & \rho \in [0, b^2) \\ +\infty & \text{otherwise} \end{cases}$$

convex duality $a(\beta) \doteq \sup_{\rho \in \mathbb{R}} \{\rho \, \beta - \phi(\rho)\} \quad \forall \, \beta \in \mathbb{R}_{\geq \phi'(0)}$

$$\Phi(\rho) = \sup_{\beta \geq \phi'(0)} \{\rho \, \beta - a(\beta)\} = \sup_{\alpha \geq -\phi(0)} \left\{a^{-1}(\alpha) \, \rho - \alpha\right\}$$

Relaxation $$l(x) = \sum_{i=1}^{N} \nu_i \sup_{\alpha_i \geq -\phi(0)} \left\{a^{-1}(\alpha_i) \, q_i(x) - \alpha_i\right\} \qquad \forall \, x \in \mathbb{R}^n$$

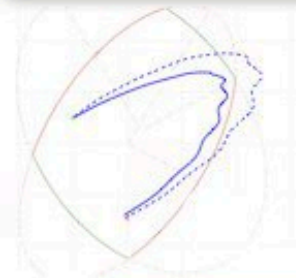
Approx. relaxation $$l^M(x) = \sum_{i=1}^{N} \nu_i \sup_{\alpha_i \in [-\phi(0), \, M]} \left\{a^{-1}(\alpha_i) \, q_i(x) - \alpha_i\right\}$$

with optimizers $$\hat{\alpha}_i^*(x) = \arg\max_{\alpha_i \geq -\phi(0)} \left\{a^{-1}(\alpha_i) \, q_i(x) - \alpha_i\right\} = \begin{cases} a \circ \phi' \circ q_i(x), & q_i(x) < b^2 \\ +\infty, & q_i(x) \geq b^2 \end{cases}$$

$$\hat{\alpha}_i^{M*}(x) = \operatorname{argmax}_{\alpha_i \in [-\phi(0), M]} \left\{a^{-1}(\alpha_i) \, q_i(x) - \alpha_i\right\} = \begin{cases} a \circ \phi' \circ q_i(x) & q_i(x) \in [0, \hat{\rho}(M)] \\ M & q_i(x) \in \mathbb{R}_{> \hat{\rho}(M)} \end{cases}$$

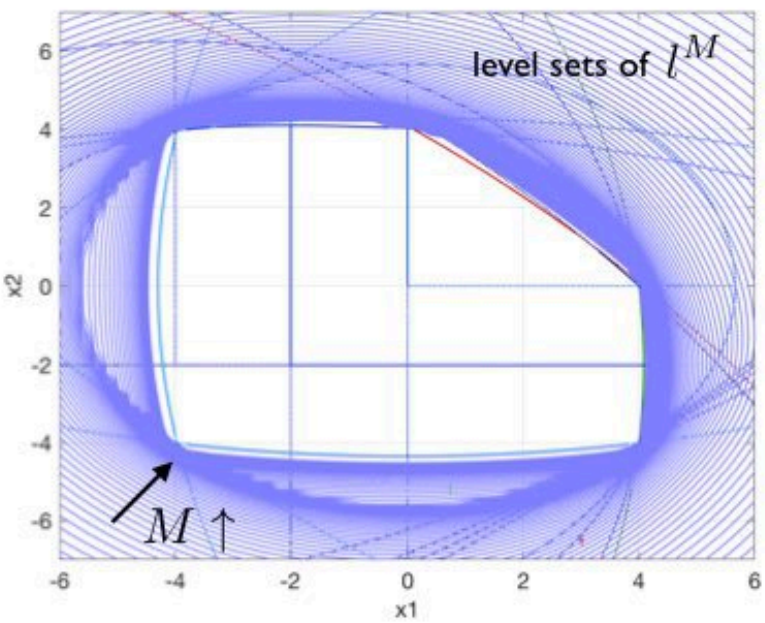
$$\hat{\rho}(M) \doteq (\phi')^{-1} \circ a^{-1}(M)$$

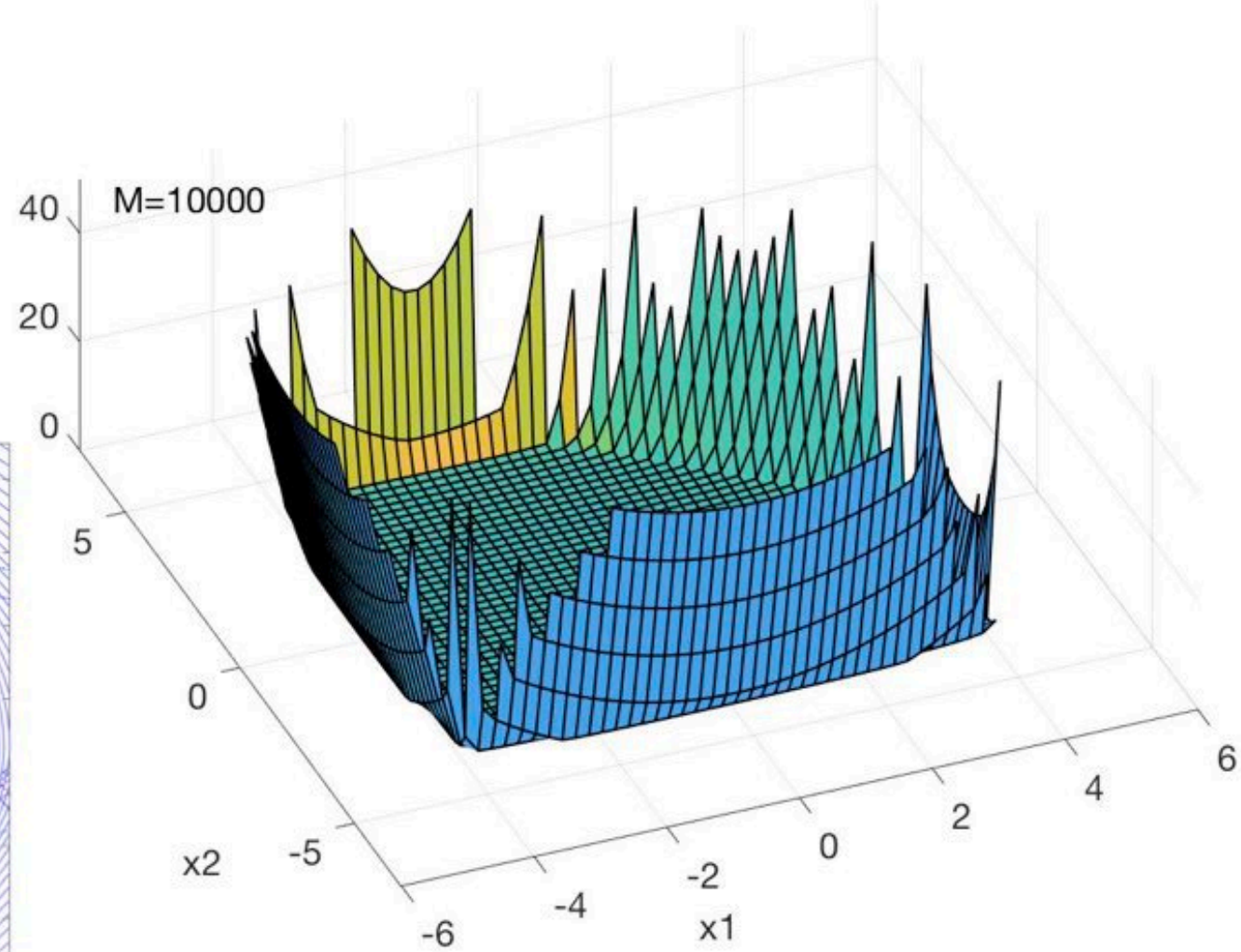
Relaxation methods



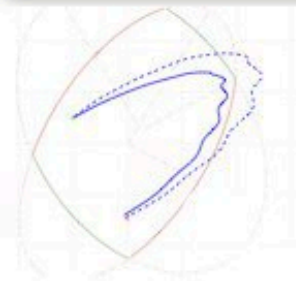
Convergence $\lim_{M\to\infty} l^M(x) = \sup_{M\geq 0} l^M(x) = l(x) \qquad \forall\, x \in \mathbb{R}^n$

$l^M(x) \quad M \in [1, 10^4]$





Relaxation methods



$$W_t(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

(value)

$$W_t^M(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t^M(x,u)$$

(approximation)

Approx. cost

$$J_t^M(x,u) = \int_t^T l^M(x_s) + \tfrac{1}{2}|u_s|^2 ds + \tfrac{1}{2}\langle x_t - z,\, P_T(x-z)\rangle$$

with approx. relaxation $\quad l^M(x) = \sum_{i=1}^N \nu_i \sup_{\alpha_i \in [-\phi(0),\, M]} \{a^{-1}(\alpha_i)\, q_i(x) - \alpha_i\}$

Game payoff

$$J_t(x,u,\alpha) = \int_t^T \nu(x_s,\alpha_s)\, ds + \tfrac{1}{2}\|u\|^2_{\mathscr{U}[t,T]} + \tfrac{1}{2}\langle x_t - z,\, P_T(x-z)\rangle$$

$$\nu(x,\hat{\alpha}) \doteq \tfrac{1}{2}\sum_{i=1}^N \nu_i\, [a^{-1}(\hat{\alpha}_i)\, q_i(x) - \hat{\alpha}_i]$$

measurable selection

Approximate game

$$W_t^M(x) = \inf_{u \in \mathscr{U}[t,T]} \sup_{\alpha \in \mathscr{A}^M[t,T]} J_t(x,u,\alpha)$$

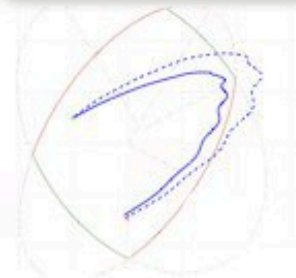
(lower value)

$$\widetilde{W}_t^M(x) = \sup_{\alpha \in \mathscr{A}^M[t,T]} \inf_{u \in \mathscr{U}[t,T]} J_t(x,u,\alpha)$$

$\alpha \in \mathscr{A}^M[0,t] \doteq C([0,t];[-\phi(0),M]^N)$

$u \in \mathscr{U}[0,t]$ (open-loop actions)

Relaxation methods



$$W_t(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

(value)

$$W_t^M(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t^M(x,u)$$

(approximation)

Approximate game

upper and lower values are equal

$$W_t^M(x) = \widetilde{W}_t^M(x) = \sup_{\alpha \in \mathscr{A}^M[t,T]} \inf_{u \in \mathscr{U}[t,T]} J_t(x,u,\alpha)$$

lower value

$u \mapsto J_t(x,u,\alpha)$ is quadratic

LQR problem

$$\widehat{W}_t^\alpha(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t(x,u,\alpha) = \frac{1}{2}\left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \hat{P}_t^\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle$$

where

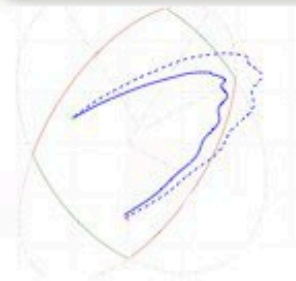
$$-\dot{\hat{P}}_s^\alpha = \hat{V}_s^\alpha - \hat{P}_s^\alpha \, \hat{P}_s^\alpha \qquad s \in (t,T)$$

$$\hat{P}_T^\alpha = \left(\begin{array}{c|c} P_T & -P_T\, z \\ \hline -(P_T\, z)' & z'\, P_T\, z \end{array}\right)$$

and

$$\hat{V}_s^\alpha \doteq \sum_{i=1}^N \nu_i \, a^{-1}([\alpha_i]_s) \left(\begin{array}{c|c} E_i & -E_i\, \bar{x}_i \\ \hline -(E_i\, \bar{x}_i)' & \bar{x}_i'\, E_i\, \bar{x}_i \end{array}\right) + \nu_i \left(\begin{array}{c|c} 0_{n\times n} & 0_n \\ \hline 0_n' & -[\alpha_i]_s \end{array}\right)$$

Relaxation methods



$$W_t(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t(x,u)$$

$$W_t^M(x) \doteq \inf_{u \in \mathscr{U}[t,T]} J_t^M(x,u)$$

Approximate game

$$W_t^M(x) = \widetilde{W}_t^M(x) = \sup_{\alpha \in \mathscr{A}^M[t,T]} \inf_{u \in \mathscr{U}[t,T]} J_t(x,u,\alpha)$$

(lower value)

Optimal trajectory

incl. state feedback characterization

$$\exists\,! \qquad \hat{P}^* \in C([t,T];\Sigma^{n+1}) \cap C^1((t,T);\Sigma^{n+1})$$

$$x^* \in C([t,T];\mathbb{R}^n) \cap C^1((t,T);\mathbb{R}^n)$$

(self-adjoint matrices)

s.t.

$$\dot{x}_s^* = -\left(\begin{array}{c|c} I_n & 0_n \end{array} \right) \hat{P}_s^* \left(\frac{x_s^*}{1} \right) \qquad x_t^* = x$$

(TPBVP)

$$-\dot{\hat{P}}_s^* = \hat{V}_s^{\hat{\alpha}^{M*}(x_s^*)} - \hat{P}_s^* \, \hat{P}_s^* \qquad \hat{P}_T^* = \left(\begin{array}{c|c} P_T & -P_T\, z \\ \hline -z'\, P_T & z'\, P_T\, z \end{array} \right)$$

where

$$\hat{V}_s^{\alpha} \doteq \sum_{i=1}^{N} \nu_i\, a^{-1}([\alpha_i]_s) \left(\begin{array}{c|c} E_i & -E_i\, \bar{x}_i \\ \hline -(E_i\, \bar{x}_i)' & \bar{x}_i'\, E_i\, \bar{x}_i \end{array} \right) + \nu_i \left(\begin{array}{c|c} 0_{n\times n} & 0_n \\ \hline 0_n' & -[\alpha_i]_s \end{array} \right)$$

$$\hat{\alpha}_i^{M*}(x) = \operatorname*{argmax}_{\alpha_i \in [-\phi(0),M]} \left\{ a^{-1}(\alpha_i)\, q_i(x) - \alpha_i \right\} = \left\{ \begin{array}{ll} a \circ \phi' \circ q_i(x) & q_i(x) \in [0, \hat{\rho}(M)] \\ M & q_i(x) \in \mathbb{R}_{>\hat{\rho}(M)} \end{array} \right.$$

Extends to more general dynamics

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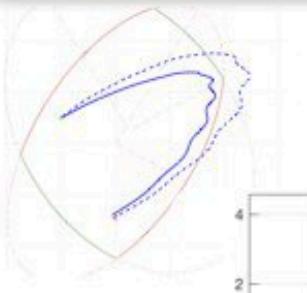
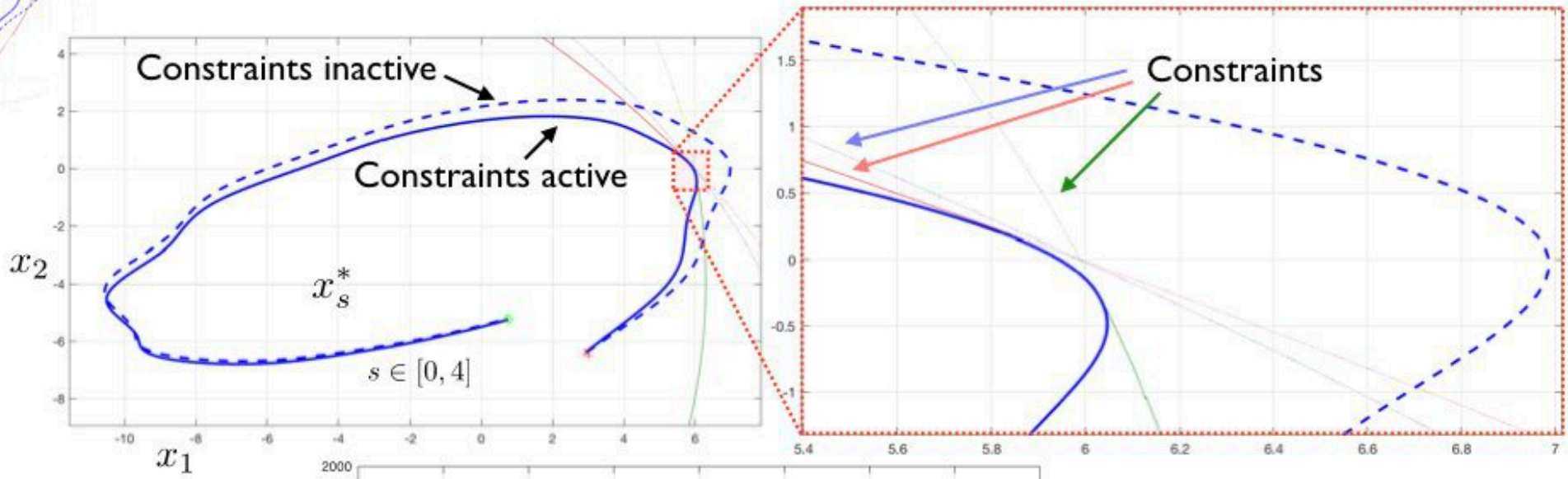
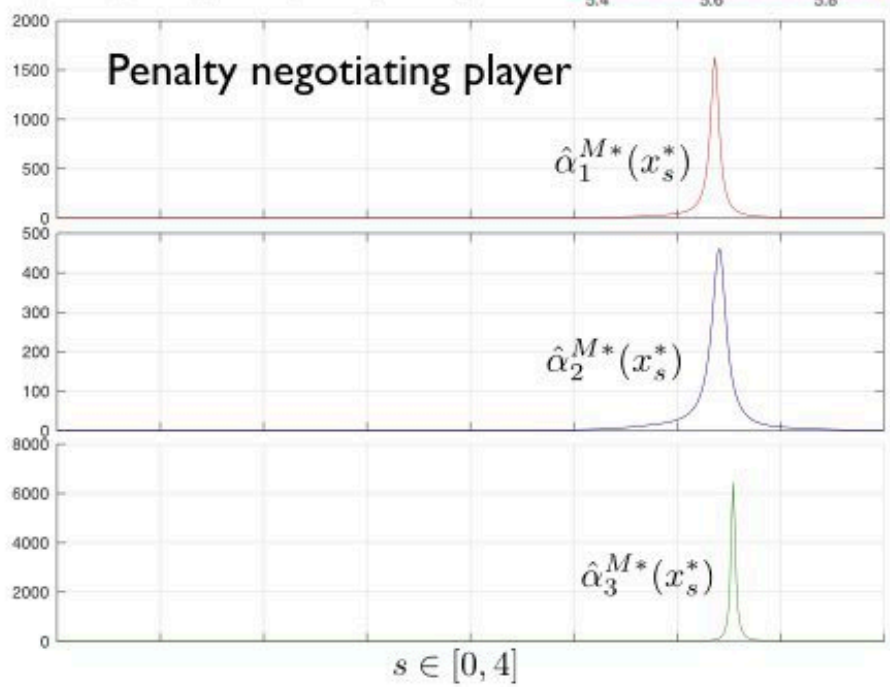
Relaxation methods



$$\dot{x}_s = A_s\, x_s + B_s\, u_s$$

$$A_s \doteq \begin{pmatrix} -1 + \frac{2}{5}\sin(\frac{16\,\pi\, s}{t}) & 2 \\ -1 & 1 \end{pmatrix},\ B_s \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$





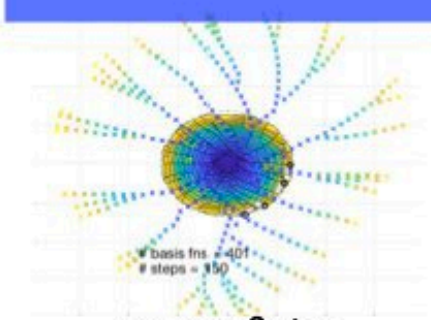
non-quadratic cost encoded via penalty negotiating player

Optimal control problem

HJB PDE $$0 = -\frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_T(x) = \Psi(x)$$

HJB PDE $$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)) \qquad W_0(x) = \Psi(x)$$

max-plus methods

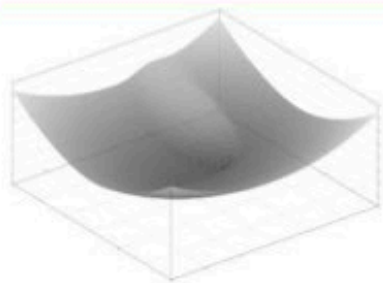


sparse finite elements

[D., McEneaney]

- error analysis
- min-plus, *stat* duality
- Hessian adaption
- infinite dimensions

optimization methods

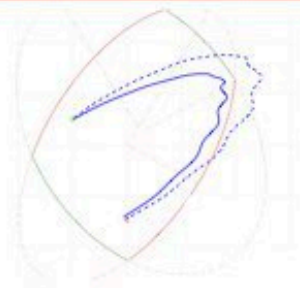


optimization over characteristics

[D., Yegorov, Grune]

- higher dimensions
- generalization to *stat*
- fundamental solutions
- interpolation & sparse grids

relaxation methods



decomposition into LQR problems

[D., Basco, McEneaney]

- infinite horizon
- non-convex constraints
- fundamental solutions
- infinite dimensions

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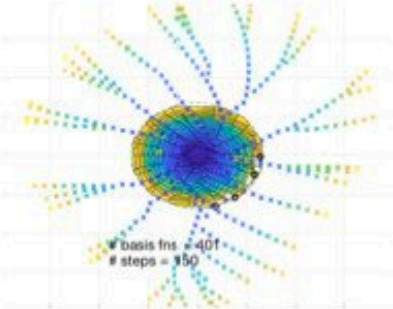
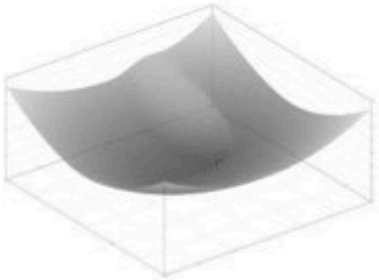
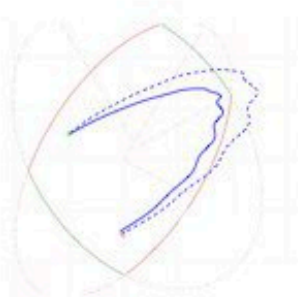
"Sparse approximation methods for optimal control"







Selected references

- [D.] "An adaptive max-plus eigenvector method for continuous time optimal control problems." In Numerical Methods for Optimal Control Problems, Ed. Falcone et al., Springer, 2018.
- [Yegorov, D.] "Perspectives on characteristics based curse-of-dimensionality-free numerical approaches to solving Hamilton Jacobi equations." AMO, 2018.
- [D., McEneaney, Yegorov] "Exploiting characteristics in stationary action problems." SIAM CT'19.
- [Yegorov, D., Grune] "Synthesis of control Lyapunov functions and stabilizing feedback strategies using exit time optimal control." arXiv:1906.02703, 2019.
- [D., McEneaney, Cantoni] "Game representations for state constrained continuous time linear regulator problems." arXiv:1904.05552, 2019.
- [D., McEneaney] "Verifying fundamental solution groups for lossless wave equations via stationary action and optimal control." arXiv: 1906.03592, 2019.
- [D., McEneaney] "Verification of stationary action trajectories via optimal control." Submitted to ACC 2020.
- [D., McEneaney] "A min-plus fundamental solution semigroup for a class of approximate infinite dimensional optimal control problems." Submitted to ACC 2020.
- [McEneaney, D.] "Conversion of a class of stochastic control problems to fundamental-solution deterministic control problems." Submitted to ACC 2020.
- [Basco, D.] "A two-player representation for a class of infinite horizon control problems under state constraints." Submitted to ACC 2020.

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