Taming the Curse of Dimensionality in HJB by Polynomial Approximation & Tensor Decomposition

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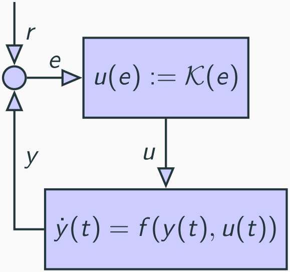
Optimal Feedback Control of Nonlinear Dynamics

Infinite horizon optimal control:



$$\underset{u(\cdot)\in\mathcal{U}}{\text{minimize}} \quad \mathcal{J}(u(\cdot), \mathbf{x}) := \int_0^\infty \ell(\mathbf{y}(t), u(t))\, dt$$

$$\text{subject to} \quad \dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t)) + \mathbf{g}u(t)$$

$$\mathbf{y}(0) = \mathbf{x} \in \mathbb{R}^n$$

Dynamic Programming: the value function

$$V(\mathbf{x}) := \inf_{u\in\mathcal{U}} \mathcal{J}(u, \mathbf{x})$$

satisfies the Hamilton-Jacobi-Bellman equation

$$\min_{u\in U} \left[(\mathbf{f}(\mathbf{x}) + \mathbf{g}u)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}, u)\right] = 0\,, \quad V(\mathbf{0}) = 0\,.$$

The optimal control is given in feedback form:

$$u^*(\mathbf{x}) = \mathcal{K}(\mathbf{x}) := \underset{u\in U}{argmin} \left[(\mathbf{f}(\mathbf{x}) + \mathbf{g}u)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}, u)\right]\,.$$

The Hamilton-Jacobi-Bellman Equation

Given $\lambda \geq 0$, $\mathbf{f}(\mathbf{x}, u)$ and $\ell(\mathbf{x}, u) \in C^1(\Omega)$, find $V(\mathbf{x})$ satisfying

$$\lambda V(\mathbf{x}) + \min_{u \in U} \left[\mathbf{f}(\mathbf{x}, u)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}, u) \right] = 0 \,, \quad \text{in } \Omega \subset \mathbb{R}^n$$

- Nonlinear PDE (minimization over U), non-divergence form.
- n-dimensional dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \Rightarrow n$-dimensional HJB.
- Globally optimal controls in feedback form: $u^*(\mathbf{x}) = \mathcal{K}(\mathbf{x})$.
- Controlling nonlinear mechanical systems/aircrafts ($d \leq 12$).
- High-dimensional HJB for the control of nonlinear PDEs?
  - Nonlinear advection/reaction: $y_t = \epsilon \Delta y + y \cdot \nabla y + gu$
  - Fokker-Planck eqns: $y_t = \epsilon \Delta y + \nabla \cdot (yu)$

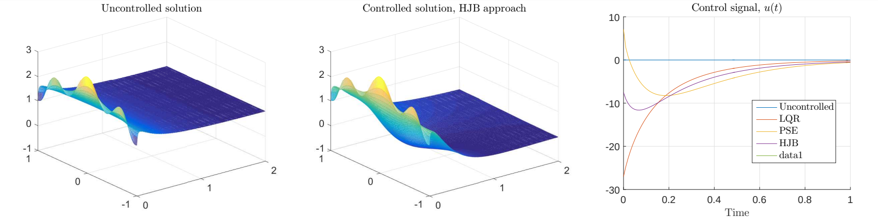
Diffusion with Polynomial Reaction

System dynamics (Allen-Cahn equation):

$$X_t(\xi,t) = X_{\xi\xi}(\xi,t) + X(\xi,t) - X(\xi,t)^3 + \mathcal{I}_\omega(\xi)u(t)\,,$$
$$X_\xi(-1,t) = 0\,, \quad X_\xi(1,t) = 0\,, \quad X(\xi,0) = cos(2\pi\xi)cos(\pi\xi) + 2.$$

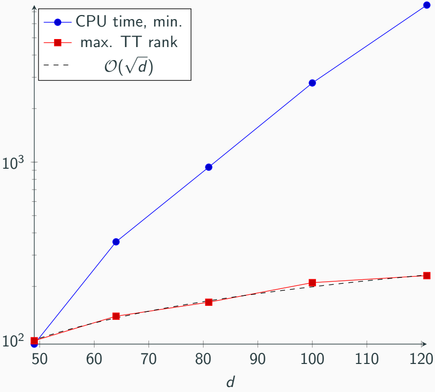
- Dynamics approximated by 14D nonlinear ODE system.
- Uncontrolled dynamics are attracted by $X = 1$.

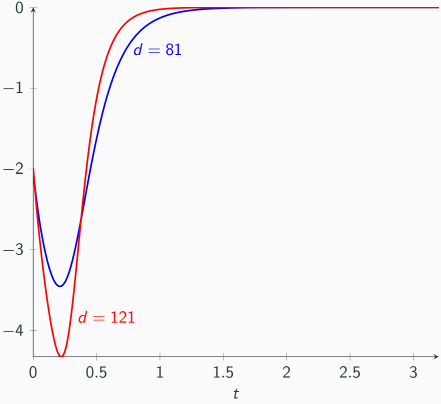
	Uncontrolled	LQR	PSE	HJB
$J(u, X_o)$	$+\infty$	10.17	9.69	8.85



D.K. and K. Kunisch, *Polynomial approximation of high-dimensional Hamilton-Jacobi-Bellman equations and applications to feedback control of semilinear parabolic PDEs*, SISC, 2018.

2D Allen-Cahn Equation , 121 DoFs





S. Dolgov, D.K. and K. Kunisch, *Tensor decomposition for high-dimensional Hamilton-Jacobi-Bellman equations*, arxiv:1908.01533, 2019.

The Curse of Dimensionality

The curse of dimensionality (artist's impression)

The Curse of Dimensionality

A non-exhaustive list:

- HJB in $\mathbb{R}^d$ approximated with a tensorial grid : N^d DoF (!).
- Physical space problems $d \leq 3$: high-order methods, DD, adaptivity.
- For $d \leq 8$: sparse grids (Garcke et al. 13'/16', Kang and Wilcox 15').
- Max-plus algebra (McEneaney 06').
- Representation formulas (Osher-Darbon 16', Yegorov-Dower 18').
- High-dimensional HJB and ML: Reinforcement Learning (Bertsekas NDP 90's), Deep BSDE (E-Han-Jentzen 17'), DGM (Sirignano-Spiliopoulos 18'), DNN+Open-loop (Nakamura-Zimmerer-Gong-Kang 19').
- Tensor calculus for linear HJB (Kappen 05'/Todorov 06', Horowitz et al. 14').

Towards HJB for PDE control:

- Coupling with MOR (Kunisch et a. 04', Alla and Falcone 13').
- Model Predictive Control and local HJB variants (Alla-Falcone-Saluzzi 18').
- Structural properties: LQR, State-Dependent Riccati Equation.

The Hamilton-Jacobi-Bellman Equation

The value function $V(\mathbf{x}) = \inf_{u \in \mathcal{U}} J(u, \mathbf{x})$ satisfies

$$\min_{u \in U} \left[(\mathbf{f}(\mathbf{x}) + \mathbf{g}u)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}) + \|u\|_R^2 \right] = 0\,, \quad V(0) = 0\,.$$

The optimal feedback for the unconstrained case ($U = \mathbb{R}$) leads to

$$u^*(\mathbf{x}) = K(\mathbf{x}) = -\frac{1}{2} R^{-1} \mathbf{g}^T \nabla V(\mathbf{x})\,,$$

$$\mathbf{f}(\mathbf{x})^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}) - \frac{1}{4} \nabla V^T(\mathbf{x}) \mathbf{g} R^{-1} \mathbf{g}^T \nabla V(\mathbf{x}) = 0\,, \quad V(0) = 0\,.$$

- Solving nonlinear HJB: policy iteration (Howard's alg.), Newton method, Newton-Kleinman iteration for Riccati equations.
- Continuous time setting.

A. Alla, M. Falcone and D. K., *An efficient policy iteration algorithm for dynamic programming equations*, SIAM J. Sci. Comput. 37(1)(2015), A181–A200.

R. W. Beard, G. N. Saridis, and J. T. Wen. *Approximate solutions to the Time-Invariant Hamilton-Jacobi-Bellman equation,* J. Optim. Theory Appl. 96(3)(1998) 589–626.

Successive Approximation Algorithm

Algorithm 1 Succesive Approximation

Input: $tol > 0$, stabilizing control $u^0(\mathbf{x})$
Result: $V^\infty(\mathbf{x}), u^\infty(\mathbf{x})$
while $\|V^n - V^{n+1}\| \geq tol$ **do**
   1. Solve for $V^{n+1}(\mathbf{x})$

$$(\mathbf{f}(\mathbf{x}) + \mathbf{g}u^n)^T \nabla V^{n+1}(\mathbf{x}) + \ell(\mathbf{x}) + \|u^n(\mathbf{x})\|_R^2 = 0\,.$$

   2. Update $u^{n+1}(\mathbf{x}) = -\frac{1}{2}R^{-1}\mathbf{g}^T\nabla V^{n+1}(\mathbf{x})$.
   3. $n = n + 1$.

end while

The optimal feedback $u^\infty(\mathbf{x})$ requires an stabilizing control $u^0(\mathbf{x})$.

- Undiscounted: $u_0(\mathbf{x})$ must be asymptotically stabilizing (in Ω).
- Path-following/continuation with respect to discount λ

$$\lambda V^{n+1}(\mathbf{x}) + (\mathbf{f}(\mathbf{x}) + \mathbf{g}u^n)^T \nabla V^{n+1}(\mathbf{x}) + \ell(\mathbf{x}) + \|u^n(\mathbf{x})\|_R^2 = 0\,.$$

Galerkin Approximation of the GHJB Equation

- Given $u^n(\mathbf{x})$, we solve the linear Generalized HJB equation

$$(\mathbf{f}(\mathbf{x}) + \mathbf{g}u^n)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}) + \|u^n\|_R^2 = 0\,.$$

- With $\{\phi_j(\mathbf{x})\}_{j=1}^{\infty}$ a complete set of *d*-dimensional polynomial basis functions, we approximate

$$V(\mathbf{x}) \approx \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}).$$

- The Galerkin residual eqn. reads

$$\langle (f(\mathbf{x}) + \mathbf{g}u^n)^T \nabla V(\mathbf{x}) + \ell(\mathbf{x}) + \|u^n\|_R^2, \phi_i(\mathbf{x}) \rangle_{L^2(\Omega)} = 0\,, \quad \forall\, i \le N\,,$$

where $\Omega \subset \mathbb{R}^d$ is a subset of the state space.

Galerkin Approximation of the GHJB Equation

- We assume that u^n $(n > 0)$, is expressed in the form

$$u^n(\mathbf{x}) = -\frac{1}{2}R^{-1}\mathbf{g}^T\nabla V^n(\mathbf{x}) = -\frac{1}{2}R^{-1}\mathbf{g}^T\sum_{j=1}^{N} c_j^n \nabla\phi_j(\mathbf{x}) \,.$$

$$\|u^n\|_R^2 = \frac{1}{4}R^{-1}(\mathbf{g}^T\nabla V^n(\mathbf{x}))^2 = \frac{1}{4}R^{-1}\left(\sum_{j=1}^{N} c_i^n \mathbf{g}^T\nabla\phi_j(\mathbf{x})\right)^2 \,,$$

  leading to

$$\langle\|u^n\|_R^2, \phi_i\rangle_{L^2(\Omega)} = (c^n)^T\mathbf{U}_{(i,\bullet)}c^n \,,$$

  where $\mathbf{U} \in \mathbb{R}^{N\times N\times N}$ is given by

$$\mathbf{U}_{(i,j,k)} = \langle(\mathbf{g}^T\nabla\phi_j)(\mathbf{g}^T\nabla\phi_k), \phi_i\rangle_{L^2(\Omega)} \,.$$

- Once every term has been expanded, we are left with a dense linear system for V^{n+1}

$$A(c^n)c^{n+1} = b(c^n) \,.$$

Computation of d-dimensional Integrals

- The free dynamics $f(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^d$ are $\mathcal{C}^1$ and separable in every coordinate $f_i(\mathbf{x})$

$$f_i(\mathbf{x}) = \sum_{j=1}^{N_f} \prod_{k=1}^{d} \mathcal{F}_{(i,j,k)}(x_k)\,,$$

where $\mathcal{F}(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}^{d \times N_f \times d}$ is a tensor-valued function. A semilinear PDE of the form

$$\partial_t X(\xi, t) = \partial_{\xi\xi} X(\xi, t) + \mathcal{N}(X, \partial_\xi X) + g(\xi) u(t)$$

when is semi-discretized in space and/or reduced, leads to a separable representation

$$\dot{X}(t) = AX(t) + N(X(t)) + gu(t)\,.$$

Computation of d-dimensional Integrals

- Construction of a separable set of basis functions, e.g., tensor product of one-dimensional basis functions,

$$\phi_i(\mathbf{x}) = \prod_{j=1}^{d} \phi_i^j(x_j)\,.$$

- Further reduction of the basis: total degree, Smolyak grids, hyperbolic cross approximation.
- The calculation of the d-dimensional integrals is reduced to the product of d, one-dimensional integrals.

The term $\langle \|u^0\|_R^2, \phi_i \rangle_{L^2(\Omega)}$ requires the computation of the product

$$\langle (\mathbf{g}^T \nabla \phi_j)(\mathbf{g}^T \nabla \phi_k), \phi_i \rangle_{L^2(\Omega)} = \mathbf{g}^T \tilde{\mathbf{U}}_{(\mathcal{I},\bullet)} \mathbf{g}\,, \quad \mathcal{I} = (i, j, k)\,,$$

with $\tilde{\mathbf{U}} \in \mathbb{R}^{N\times N\times N\times d\times d}$ given by $\tilde{\mathbf{U}}_{(\mathcal{I},l,m)} := \langle \partial_{x_l}\phi_j \partial_{x_m}\phi_k, \phi_i \rangle_{L^2(\Omega)}\,.$

Cardinality of the Total Degree Basis

- # basis polynomials of M-th order in $\mathbb{R}^d$:

$$\#P_M^d = \sum_{m=1}^{M} \binom{d+m-1}{m}$$

- Odd-degree polynomials can be neglected if $f(-x) = -f(x)$.

	Full monomial basis				Even-degree monomials			
$d \backslash M$	2	4	6	8	2	4	6	8
6	27	209	923	3002	21	147	609	1896
8	44	494	3002	12869	36	366	2082	8517
10	65	1000	8007	43757	55	770	5775	30085
12	90	1819	18563	125969	78	1443	13819	89401
14	119	3059	38759	319769	105	2485	29617	233107

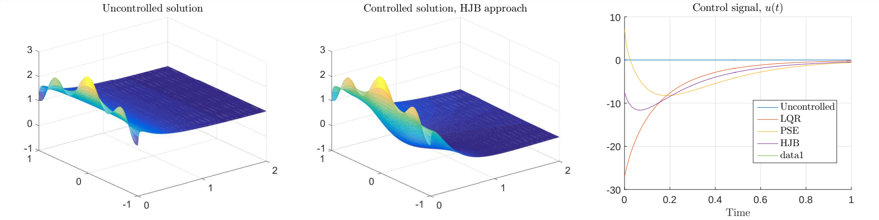
Cardinality of the multdimensional basis

System dynamics (Allen-Cahn equation):

$$X_t(\xi,t) = X_{\xi\xi}(\xi,t) + X(\xi,t) - X(\xi,t)^3 + \mathcal{I}_\omega(\xi)u(t)\,,$$
$$X_\xi(-1,t) = 0\,,\quad X_\xi(1,t) = 0\,,\quad X(\xi,0) = cos(2\pi\xi)cos(\pi\xi) + 2.$$

- HJB solver: 14 dimensions, monomial basis of order 4.
- Uncontrolled dynamics are attracted by $X = 1$.

	Uncontrolled	LQR	PSE	HJB
$J(u, X_o)$	$+\infty$	10.17	9.69	8.85



D.K. and K. Kunisch, *Polynomial approximation of high-dimensional Hamilton-Jacobi-Bellman equations and applications to feedback control of semilinear parabolic PDEs*, SISC, 2018.

Nonlinear $\mathcal{H}_\infty$ Control and the Isaacs Equation

$$\min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} J(u,w,\mathbf{x}) := \int_0^\infty \ell(\mathbf{x}(t)) + \|u(t)\|_R^2 - \gamma^2 \|w(t)\|_P^2 \, dt$$

subject to the dynamical constraint

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t)) + \mathbf{g}_1 w(t) + \mathbf{g}_2 u(t)\,, \quad \mathbf{y}(0) = \mathbf{x}\,.$$

- The perturbation $w(t): \mathbb{R}_+ \to W \subset \mathbb{R}^m$, and $\|w\|_P^2 = Pw^2$.
- The parameter γ is larger than the $\mathcal{H}_\infty$-norm of the system.

In the unconstrained case the value satisfies a HJ-Isaacs equation:

$$\nabla V_\gamma(\mathbf{x})^t \mathbf{f}(\mathbf{x}) + \frac{1}{4} \nabla V_\gamma(\mathbf{x}) Q(\mathbf{x}) \nabla V_\gamma(\mathbf{x})^t + \ell(\mathbf{x}) = 0\,,$$

with

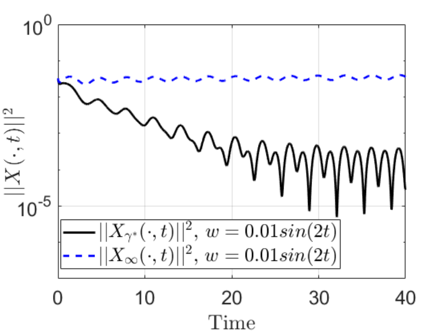
$$Q(\mathbf{x}) := \mathbf{g}_1(\mathbf{x}) P^{-1} \gamma^{-2} \mathbf{g}_1(\mathbf{x})^t - \mathbf{g}_2(\mathbf{x}) R^{-1} \mathbf{g}_2(\mathbf{x})^t$$

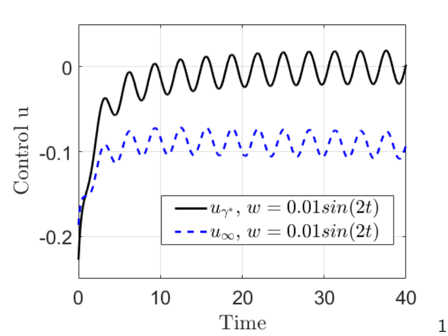
Testing the Performance of the $\mathcal{H}_\infty$ Control

Allen-Cahn equation with uncertain dynamics:

$$\begin{aligned}
\partial_t X(\xi,t) &= \sigma \partial_{\xi\xi} X(\xi,t) - X(\xi,t)^3 + X(\xi,t)w(t) + \chi_{\omega_2}(\xi)u(t)\,,\\
\partial_\xi X(\xi_l,t) &= \partial_\xi X(\xi_r,t) = 0\,,\\
X(\xi,0) &= \kappa(\xi-1)^2(\xi+1)^2\,.
\end{aligned}$$

- Differences between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ closed-loops.
- Disturbance set as $w = 1$. Destabilizing for $\mathcal{H}_2$.





The Tensor Calculus Approach

Solving the linear high-dimensional PDE

$$(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_{k-1}(\mathbf{x}))^\top DV_k(\mathbf{x}) + \ell(\mathbf{x}) + \gamma u_{k-1}^2(\mathbf{x}) = 0\,.$$

- Tensorial finite element discretization: quickly intractable for $d > 3$.
- Polynomial basis with TD: mitigates CoD for $d \leq 15$ and $M \leq 6$.
- The linear HJB equation can be solved through tensor calculus.

Tensorised Legendre polynomials of maximal individual degree $n-1$,

$$\mathcal{V}_n = \operatorname{span}\{\Phi_{\mathbf{i}}(\mathbf{x}) := \phi_{i_1}(x_1)\cdots\phi_{i_d}(x_d), \quad i_k = 0,\dots,n-1, \quad k = 1,\dots,d\}$$

$\phi_{i_k}(x_k)$: univariate Legendre polynomials of degree i_k, $\mathbf{i} = (i_1,\dots,i_d)$.

$$V(x_1,\dots,x_d) \approx \sum_{j_1,\dots,j_d=0}^{n-1} \mathbf{v}(j_1,\dots,j_d)\Phi_{j_1,\dots,j_d}(\mathbf{x}),$$

We solve a system of n^d Galerkin equations in n^d unknowns of $\mathbf{v}$,

$$\sum_{\mathbf{j}} \left\langle \Phi_{\mathbf{i}},\ (\mathbf{f} + \mathbf{g}\check{u})^\top D\Phi_{\mathbf{j}} \right\rangle_{L^2(\Omega)} \mathbf{v}(\mathbf{j}) = -\left\langle \Phi_{\mathbf{i}},\ \ell + \gamma\check{u}^2 \right\rangle_{L^2(\Omega)}\,.$$

The Tensor Calculus Approach

- Accuracy: standard univariate polynomial approximation theory.
  $V(\mathbf{x}) \in C^p(\Omega)$: error decays with an $\mathcal{O}(n^{-p})$ rate.
- Difficulty: we're back again to n^d DoF, for a linear problem.

The Tensor Train format (Oseledets 2001):

$$\mathbf{v}(i_1, \ldots, i_d) \approx \tilde{\mathbf{v}}(i_1, \ldots, i_d) := \sum_{\alpha_0, \ldots, \alpha_d = 1}^{r_0, \ldots, r_d} \mathbf{v}^{(1)}_{\alpha_0, \alpha_1}(i_1) \mathbf{v}^{(2)}_{\alpha_1, \alpha_2}(i_2) \cdots \mathbf{v}^{(d)}_{\alpha_{d-1}, \alpha_d}(i_d).$$

- $\mathbf{v}^{(k)}$ are 3-dim. tensors (TT blocks). $r_0, \ldots, r_d$: TT ranks.
- $r_0 = r_1 = \cdots = r_d = 1 \Rightarrow$ complete separation.
- Similar to SVD in 2d. TT decomposition cost: $d - 1$ SVD's.
- Total DoF: dnr^2. Need to adjust n and r for efficiency.

Theoretical analysis with TT for functions:

$$V(\mathbf{x}) \approx \widetilde{V}(\mathbf{x}) := \sum_{\alpha_0, \ldots, \alpha_d = 1}^{r_0, \ldots, r_d} v^{(1)}_{\alpha_0, \alpha_1}(x_1) \cdots v^{(d)}_{\alpha_{d-1}, \alpha_d}(x_d) \, .$$

The Tensor Calculus Approach

Bounds on r_k for the LQ case [DK, Dolgov, and Kunisch 19']:

$$r_k(\widetilde{V}) \leq \min\left((M + r_b)\left(\log \frac{1}{\varepsilon} + C\right)^{7/2},\ \min(k, d-k)\right) + 2\,.$$

System: $A = \left[\left\langle \Phi_{\mathbf{i}},\ (\mathbf{f} + \mathbf{g}\check{u})^\top D\Phi_{\mathbf{j}} \right\rangle_{L^2(\Omega)}\right]_{\mathbf{i},\mathbf{j}}$, $\mathbf{b} = \left[-\left\langle \Phi_{\mathbf{i}},\ \ell + \gamma \check{u}^2 \right\rangle_{L^2(\Omega)}\right]_{\mathbf{i}}$

Quadrature rules:

$$\mathbf{f}_p(\mathbf{i}) = \int f_p(\mathbf{x})\Phi_{\mathbf{i}}(\mathbf{x})d\mathbf{x} \approx \sum_{j_1,\ldots,j_d=1}^{m} w_{j_1}\cdots w_{j_d} f_p(x_{j_1},\ldots,x_{j_d})\Phi_{i_1,\ldots,i_d}(x_{j_1},\ldots,x_{j_d})\,.$$

Assume a TT decomposition of $\hat{\mathbf{f}}_p(j_1,\ldots,j_d) = f_p(x_{j_1},\ldots,x_{j_d})$:

$$f_p(x_{j_1},\ldots,x_{j_d}) \approx \sum_{\beta_0,\ldots,\beta_d=1}^{R_0,\ldots,R_d} \mathbf{f}^{(1)}_{p,\beta_0,\beta_1}(j_1)\cdots \mathbf{f}^{(d)}_{p,\beta_{d-1},\beta_d}(j_d).$$

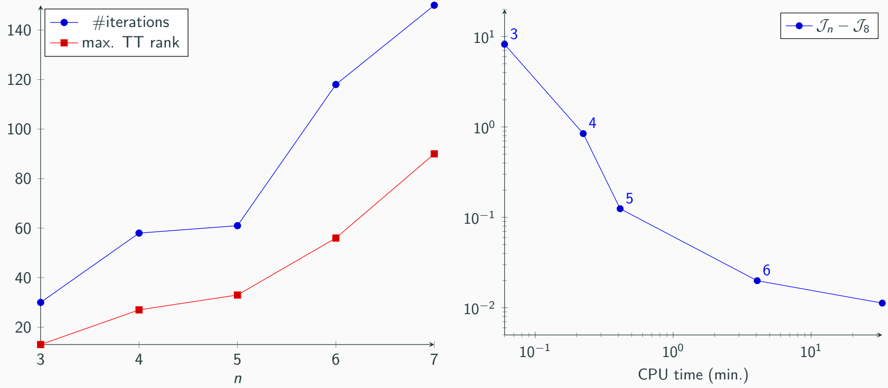
The Alternating Linear Scheme (Holtz et al. 12')

$$V_{\neq k}(i_1,\ldots,i_d;\ \alpha_{k-1},j_k,\alpha_k) = \sum_{\substack{\alpha_0,\ldots,\alpha_{k-2},\\ \alpha_{k+1},\ldots,\alpha_d}} \mathbf{v}^{(1)}_{\alpha_0,\alpha_1}(i_1)\cdots\mathbf{v}^{(k-1)}_{\alpha_{k-2},\alpha_{k-1}}(i_{k-1}) \cdot \delta(i_k,j_k)\cdot \mathbf{v}^{(k+1)}_{\alpha_k,\alpha_{k+1}}(i_{k+1})\cdots\mathbf{v}^{(d)}_{\alpha_{d-1},\alpha_d}(i_d).$$

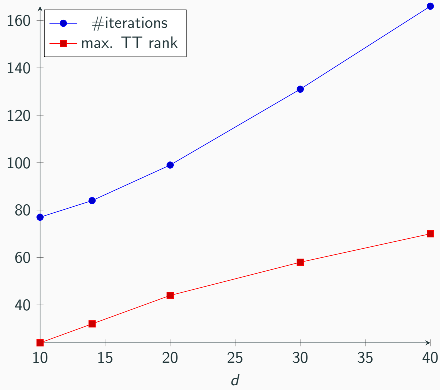
Algorithm 2 Policy update with the ALS iteration

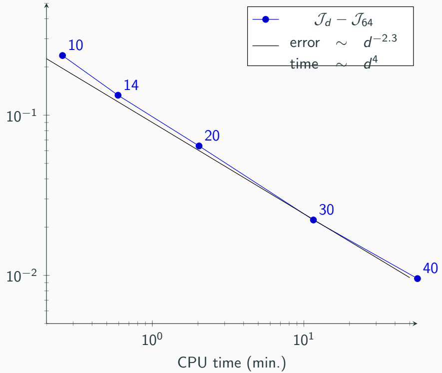
Initial value tensor $\mathbf{v}$, shift $\mu > 0$, stopping threshold $\delta > 0$, $\check{\mathbf{v}} = 0$.
while $\|\mathbf{v} - \check{\mathbf{v}}\|_2 > \delta\|\mathbf{v}\|_2$ **do** {Policy iteration}
    Construct $A[\check{u}]$ and $\mathbf{b}[\check{u}]$.
    Orthogonalise TT blocks of $\mathbf{v}$.
    for $k = 1,\ldots,d$ **do** {ALS algorithm}
        Assemble and solve $\left(V_{\neq k}^\top A V_{\neq k} + \mu I\right)\bar{v}^{(k)} = V_{\neq k}^\top \mathbf{b} + \mu V_{\neq k}^\top \check{\mathbf{v}}$.
        Update the TT block and orthogonalise.
    end for
end while

Numbers of policy iterations and maximal TT ranks, $d = 14$



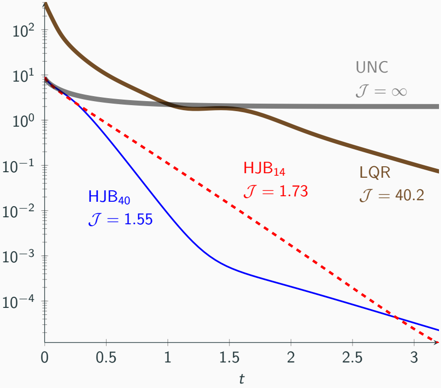
Numbers of Policy Iterations and Maximal TT Ranks, $n = 5$

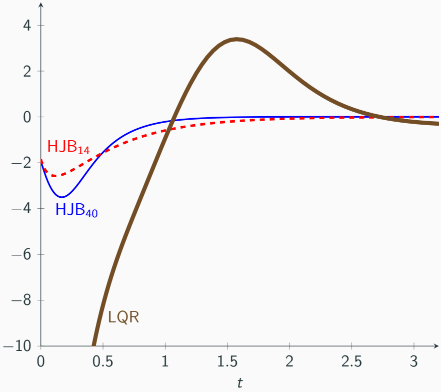




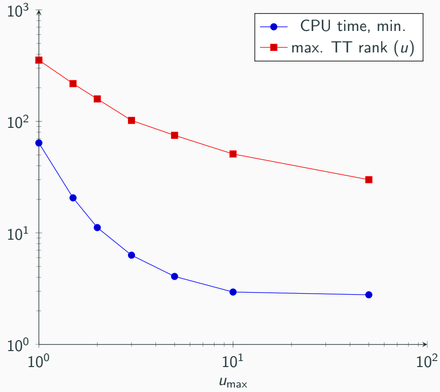
Allen-Cahn Equation $d = 40$

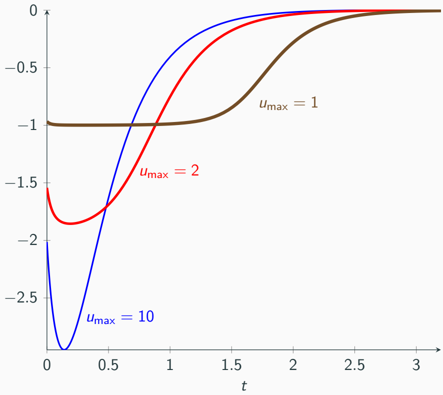
$$X_t(\xi,t) = X_{\xi\xi}(\xi,t) + X(\xi,t) - X(\xi,t)^3 + \mathcal{I}_\omega(\xi)u(t)$$





Enforcing Control Constraints Through Penalties

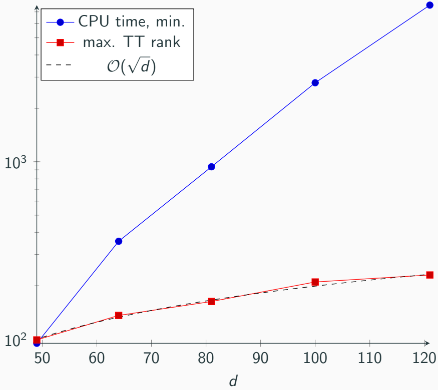


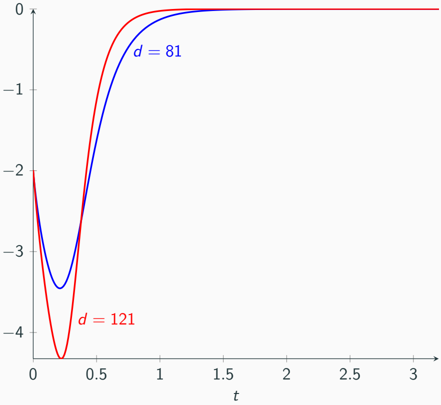


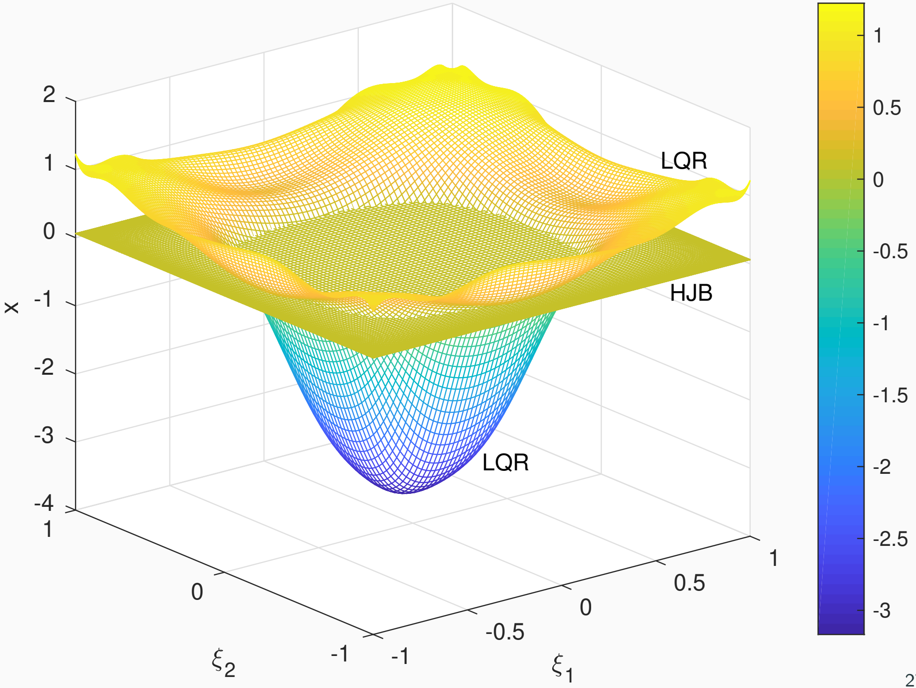
Control penalty: $2\gamma \int_0^u \mathcal{P}^{-1}(\mu) d\mu$, with $\mathcal{P}(x) = u_{\max} \cdot \tanh(x/u_{\max})$.

S.E. Lyshevski, *Optimal control of nonlinear continuous-time systems: design of bounded controllers via generalized nonquadratic functionals*, Proc. ACC 1998.

2D Allen-Cahn Equation , 121 DoFs



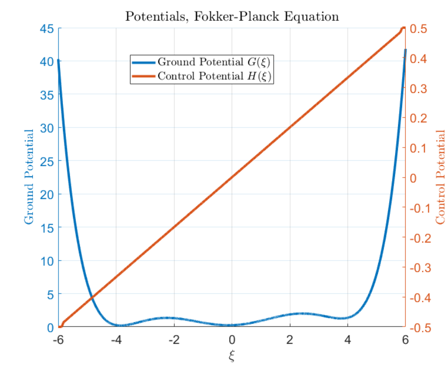


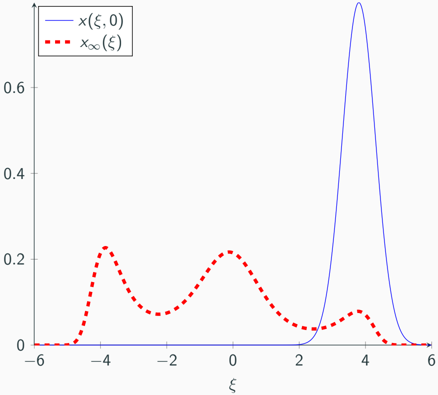

LQR
HJB
LQR
x
2
1
0
-1
-2
-3
-4
1
0
-1
ξ2
-1
-0.5
0
0.5
1
ξ1
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3

The Fokker-Planck Equation

$$\partial_t x(\xi, t) = \nu \partial_{\xi\xi} x + \partial_\xi (x \partial_\xi G) + u \partial_\xi (x \partial_\xi H), \quad \xi \in \Omega,$$

$$0 = [\partial_\xi x + x \partial_\xi (G + uH)]\,|_{\xi \in \partial\Omega}$$





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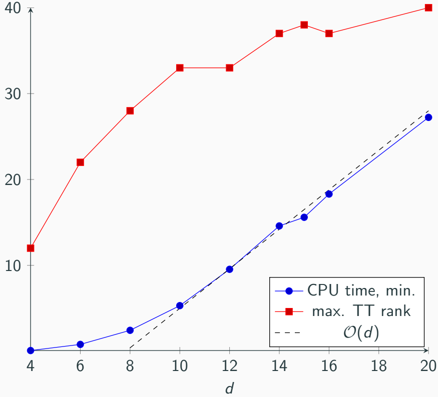

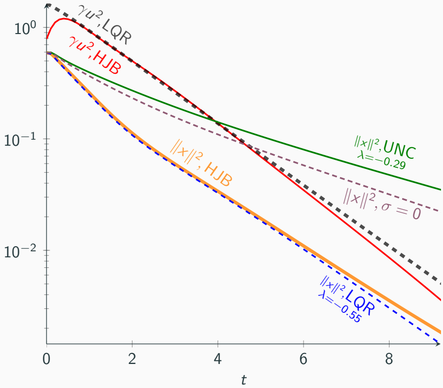

Summary:

- Spectral Methods and Policy Iteration as a core block for stationary HJB.
- Polynomial Approximation or Tensor Decompositions for high-dimensional HJB.
- Deterministic approach and analysis for problems in $+100$ dimensions.
- Inclusion of control constraints through penalties.
- Nonlinear PDEs as a class of well-structured nonlinearities

Outlook:

- Other polynomial ansatz: Smolyak grids, Hyperbolic cross.
- Convergence estimates for the TT decomposition.
- Feedback control of nonlocal PDEs: Boltzmann, McKean-Vlasov type.

Thanks for you attention!

D.K. and K. Kunisch, *Polynomial approximation of high-dimensional Hamilton-Jacobi-Bellman equations and applications to feedback control of semilinear parabolic PDEs*, SISC, 2018.

D.K., S. Kundu and K. Kunisch, *Robust feedback control of nonlinear PDEs by numerical approximation of high-dimensional Hamilton-Jacobi-Isaacs equations*, arXiv:1905.06276, 2019.

S. Dolgov, D.K. and K. Kunisch, *A Tensor Decomposition Approach for High-Dimensional Hamilton-Jacobi-Bellman Equations*, arXiv:1908.01533, 2019.