Nonlinear Systems Toolbox

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They dealt with control and estimation problems in

State Space Form.

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And there was much lamenting the gap between the linear state space theory and the linear frequency domain practice.

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In the later 1970s LINPAK and a related package called EISPAK were replaced and supplanted by LAPACK which also uses BLAS.

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MATLAB was first adopted by researchers and practitioners in control engineering, Little's specialty, but quickly spread to many other domains.

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So special purpose software is becoming available for some nonlinear systems or perhaps more precisely the Jacobi linearizations of nonlinear system, e.g. the Extended Kalman Filter.

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For linear systems we have theoretical solutions that are easily implemented numerically.

For nonlinear systems we have theoretical solutions that usually cannot be implemented numerically.

Fundamental Problems, Practitioners' Opinion

Table 1: Results of a survey by the IFAC Industry Committee on the current and future impact of PID and advanced control technologies

	Current Impact		Future Impact	
Control Technology %	**High**	**Low/No**	**High**	**Low/No**
PID control	91%	0%	78%	6%
System Identification	65%	5%	72%	5%
Estimation & filtering	64%	11%	63%	3%
Model-predictive control	**62%**	**11%**	**85%**	**2%**
Process data analytics	51%	15%	70%	8%
Fault detection & identification	48%	17%	8%	8%
Decentralized and/or coordinated control	29%	33%	54%	11%
Robust control	26%	35%	42%	23%
Intelligent control	24%	38%	59%	11%
Nonlinear control	21%	44%	42%	15%
Discrete-event systems	24%	45%	39%	27%
Adaptive control	18%	38%	44%	17%
Repetitive control	12%	74%	17%	51%
Other advanced control technology	11%	64%	25%	39%
Hybrid dynamical systems	11%	68%	33%	33%
Game theory	5%	76%	17%	52%

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I call my attempt the "Nonlinear Systems Toolbox"

Polynomial Systems

My Nonlinear Systems Toolbox (nst19) deals with polynomial systems around an operating point $x = 0, u = 0$ in continuous time

$$
\begin{aligned}
\dot{x} &= f(x,u) = Fx + Gu + f^{[2]}(x,u) + \cdots + f^{[d]}(x,u) \\
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Polynomial systems arise naturally in various contexts, e.g. chemical reactions, predator prey, or they could be the Taylor polynomials of more general smooth systems, e.g. pendula.

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Within groups the monomials are stored in lexographic order with right indices moving faster than left indices.

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There are $\mathbf{216}$ monomials of degree $\mathbf{3}$ in $\mathbf{6}$ noncommuting variables.

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Some Fundamental PDEs of Continuous Time Nonlinear Control

Hamilton-Jacobi-Bellman PDE, unkowns $c = \pi(x), u = \kappa(x)$

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0 &= \min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} \\
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Francis-Byrnes-Isidori PDE of Nonlinear Regulation, unkowns $x = \phi(w), u = \lambda(w)$

$$\begin{aligned} f(\phi(w), \lambda(w)) &= \frac{\partial \phi}{\partial w}(w) a(w) \\ h(\phi(w), \lambda(w)) &= 0 \end{aligned}$$

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Feedback Linearization PDE, unkowns $x = \phi(z), v = \alpha(z, u)$

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Kazantzis-Kravaris PDE for an observer with linear error dynamics, unknowns $x = \phi(z), \beta(h(x))$

$$f(x) \;\; = \;\; \frac{\partial \phi}{\partial z}(z)\left(F\phi^{-1}(x) + \beta(h(x))\right)$$

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Al'brekht did it for HJB PDEs in 1961 and the others were done later but the basic technique goes back Poincare and even Euler.

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Bellman's Dynamic Programming Equation is a functional equation in the unknown functions $\pi(x), \kappa(x)$.

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\pi(x) &= \min_u \{\pi(f(x,u)) + l(x,u)\} \\
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In discrete time the stochastic Dynamic Programming equation for the expected optimal cost and optimal feedback can also be computed degree by degree if the coefficients of the noise in the dynamics vanishes at the operating point.

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NST also works with polynomial matrix fields, polynomial higher order tensor fields and complex fields.

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`csq.m` **Extends a polynomial of degrees 2 through d+1 whose quadratic part is positive definite to a polynomial of degrees 2 through 2d which is a sum of squares.**

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Example Optimal Stabilization

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We call this technique Completing the Squares.

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If this is so then a shorter horizon can be used in the Model Predictive scheme.

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If they are not we conclude that the current horizon is too short and we increase it.

Example

We simulated AHMPC with two different terminal costs and terminal feedbacks. In both cases the Lagrangian was

$$\frac{0.1}{2}\left(|x|^2+|u|^2\right)$$

The first pair $\pi_f^2(x), \kappa_f^1(x)$ was found by solving the infinite horizon LQR problem obtained by taking the linear part of the dynamics around the operating point $x=0$ and the quadratic Lagrangian. Then $d=1$, $\pi_f^2(x)$ is a positive definite quadratic form and $\kappa_f^1(x)$ is linear function.

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The second pair $\pi_f^6(x), \kappa_f^5(x)$ was found using the discrete time version of Al'brekht's method to degree $d=5$. Then $\pi_f^6(x)$ is the Taylor polynomial of the optimal cost to degree 6 and $\kappa_f^5(x)$ is the Taylor polynomial of the optimal feedback to degree 5. But $\pi_f^6(x)$ is not positive definite so we completed the squares to get $\pi_f^{10}(x)$ which is positive definite.

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In all the simulations we imposed the control constraint $|u|_\infty \leq 4$ and started at $x(0) = (0.9\pi, 0.9\pi, 0, 0)$ with an initial horizon of $T = 50$ time steps.

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If the Lyapunov and feasibility conditions were comfortably satisfied over the extended horizon then the simulation was advanced one time step and the horizon T was decreased by 1 .

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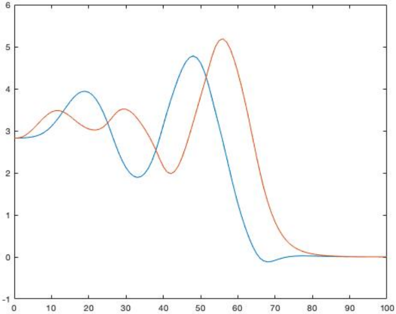
The degree $2d = 10$ terminal cost and the degree $d = 5$ terminal feedback seems to do it a little more smoothly and with shorter maximum horizon $T = 65$ versus $T = 75$ for LQR $d = 1$.

Example



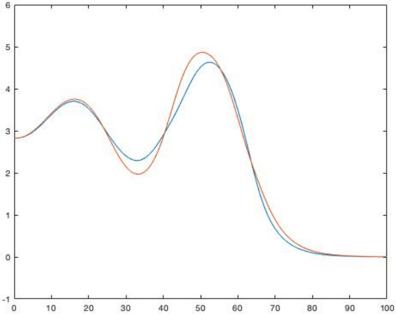


Figure: Angles, $\boldsymbol{d = 1}$ on left, $\boldsymbol{d = 5}$ on right

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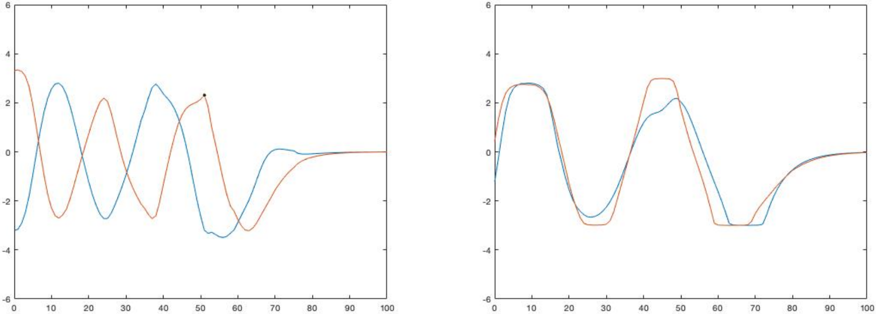

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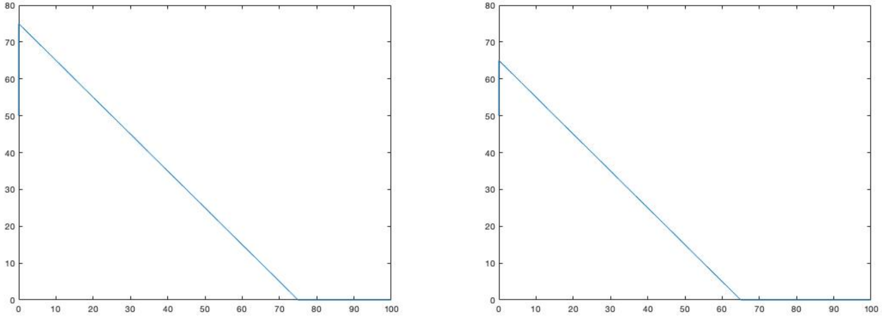

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The cpu time for the LQR terminal cost and terminal feedback was 24.56 seconds so it not clear that it is possible to control the double pendula in real time using LQR.

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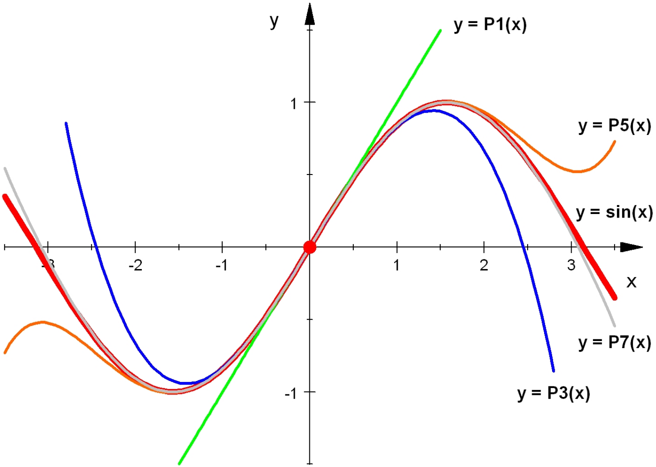

Figure: Taylor Approximations to $y = \sin x$

Rigid Body

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There are 33648 monomials of degrees 1 though 5 and 134577 monomials of degrees 2 though 6 in 18 variables.

Rigid Body

We applied Al'brekht's method to the stabilzation of rigid body. The state dimension is 12 and the control dimension is 6.

Using MATLAB' symbolic engine, `hjb_set_up.m` computed the Taylor polynomial of the dynamics to degree 3 in about 90 sec.

`hjb.m` computed the Taylor polynomials of the optimal cost and the optimal feedback to degrees 4 and 3 in less than 5 sec.

MATLAB couldn't compute the Taylor polynomial to degree 5.

`hjb.m` computed the Taylor polynomials of the optimal cost and the optimal feedback to degrees 6 and 5 in about 27 minutes.

There are 33648 monomials of degrees 1 though 5 and 134577 monomials of degrees 2 though 6 in 18 variables.

The infinity norm of the residual of the first HJB equation is $2.2666e-12$

Verification and Benchmarks

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We need more benchmark problems.

Conclusions

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Adaptive Horizon Model Predictive Control combines the Taylor polynomials of the optimal cost and feedback with Model Predictive Control to overcome their respective weaknesses.

Think Mathematically

Think Mathematically

Act Computationally

Thank You

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Questions