Conversion of Second-Order HJB PDE Problems into First-Order HJB PDE Problems

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Main Points: Computation for Nonlinear Control

- Max-plus curse-of-dimensionality-free methods (Akian, Dower, Fleming, Gaubert, Mc, Qu, et al.).
  - Exceptionally well-suited to high-dimensional, “lower-complexity” problems ($\leq$ 15-dimensional).
  - Orignally conceived for first-order HJ PDE.
  - Extension to second-order HJ PDE requires max-plus distributive property, which induces a much-higher “curse of complexity” and significantly reduced performance.
- Fundamental solution approaches (Dower, Mc, et al.).
  - A single object is generated. Solutions for varying problem data do not require re-propagation of the solution.
  - Employed in two-point boundary value problems in the classic n-body domain.
  - Semi-convex and “stat” duality are employed on the fundamental-solution object to generate solutions to particular data.

Main Points: Second-Order vs. First-Order HJ PDE

- First-order HJ PDE:
  - $0 = W_t + \min_{v \in R^n}\{f(x,v)^T W_x + L(x,v)\}$.
  - Systems with ODE dynamics.
  - Information travels along (generalized) characteristics at a finite rate, modulo shocks and rarefaction waves.
  - Nonsmooth solutions.
- Second-order HJ PDE:
  - $0 = W_t + \mathrm{tr}(AW_{xx}) + \min_{v \in R^n}\{f(x,v)^T W_x + L(x,v)\}$.
  - Systems driven by Brownian motion (SDE dynamics).
  - Information travels as an infinite-rate.
  - Nondegenerate diffusion matrix implies smooth solutions.
- In general, these two classes require significantly different numerical techniques.
- We wil convert second-order HJ PDE problems within a certain class into fundamental-solution first-order HJ PDE problems.

Section 1: Staticization

Staticization

- Need to search for stationary (static) points of action functionals.
- Terminology: Staticization, statica (analogous to minimization, minima).
- Let $\bar{y} \in \mathcal{G}_{\mathcal{Y}}$ where $\mathcal{G}_{\mathcal{Y}}$ is an open subset of a Hilbert space. We say

$$\bar{y} \in \operatorname*{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \quad \text{if} \quad \limsup_{y \to \bar{y}, y \in \mathcal{G}_{\mathcal{Y}}} \frac{|F(y) - F(\bar{y})|}{|y - \bar{y}|} = 0,$$

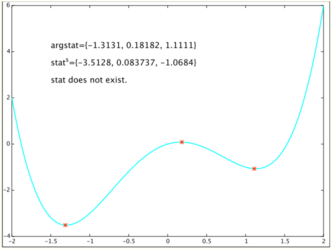
- If f is differentiable and $\mathcal{G}_{\mathcal{Y}}$ is open, then $\operatorname{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) = \{y \in \mathcal{G}_{\mathcal{Y}} \mid F_y(y) = 0\}$.
- Define set-valued $\overline{\operatorname{stat}}$ by

$$\overline{\operatorname{stat}}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \doteq \left\{ F(\bar{y}) \,\middle|\, \bar{y} \in \operatorname*{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} \{F(y)\} \right\} \text{ if } \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_{\mathcal{Y}}\} \neq \emptyset.$$

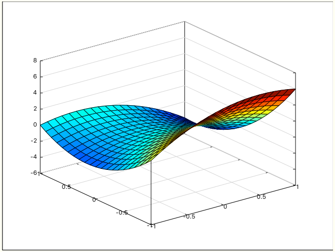
- If there exists a s.t. $\overline{\operatorname{stat}}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) = \{a\}$, then

$$\operatorname*{stat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \doteq a.$$

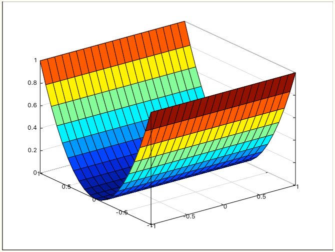
Staticization: Simple Examples



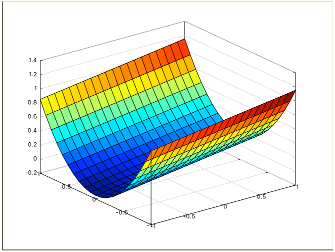
stat does not exist.



stat exists.



stat exists.



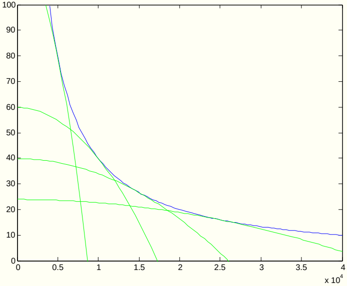
stat does not exist.

Staticization-Based Representation for the Gravitational Potential

- Classic gravitational potential energy expression for bodies at x and origin with masses m and m_0:

$$-V(x) = \frac{G m_0 m}{|x|}.$$

- Inverse norm is difficult.



- Additive inverse of potential as optimized quadratic (with $\widehat{G} \doteq (3/2)^{3/2} G$).

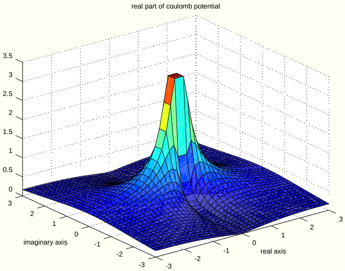
$$-V(x) = \frac{\widehat{G} m_0 m}{|x|} = \widehat{G} m_0 m \sup_{\alpha \in [0,\infty)} \left\{ \alpha - \frac{\alpha^3 |x|^2}{2} \right\}.$$

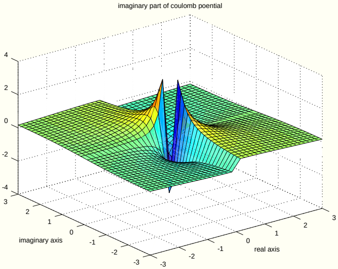
- Argument is convex cubic on $[0, \infty)$; replace sup with stat:

$$-V(x) = \widehat{G} m_0 m \operatorname*{stat}_{\alpha \in [0,\infty)} \left\{ \alpha - \frac{\alpha^3 |x|^2}{2} \right\}.$$

Coulomb potential

- Can extend Coulomb potential from $\boldsymbol{R}^n$ to $\mathbb{C}^n$.
- Additive inverse of Coulomb/gravitational potential over $\mathbb{C}$:





Staticization-based extension of Coulomb potential to $\mathbb{C}^3$

- Note that although min and max are valid only for real-valued functionals, staticization is valid for complex-valued functionals.
- The Coulomb potential, extended to $\mathbb{C}^3$ may be generated similarly to the stat representation of the gravitational potential.
- Let the Coulomb potential on $\boldsymbol{R}^3$ be given by $-V(y) = \mu_c/|y|$. Then, the extension to $x \in \mathbb{C}^3$ is (with $\hat{\mu}_c \doteq \left(\frac{3}{2}\right)^{3/2}\mu$):

$$-V(x) = \frac{\mu_c}{\sqrt{x^T x}}$$
$$= \hat{\mu} \operatorname*{stat}_{\alpha \in \mathcal{H}^+} \left[\alpha - \frac{\alpha^3 (x^T x)}{2} \right],$$

where

$$\mathcal{H}^+ \doteq \left\{ \alpha = re^{i\theta} \,|\, r > 0,\, \theta \in (-\pi/2, \pi/2] \right\}.$$

Legendre Transform and Stat-Quad Duality

- **Stat-duality (Legendre):** $\mathcal{A}, \mathcal{B}$ open; $\phi \in C^1(\mathcal{A}; \boldsymbol{R})$; $[D\phi]^{-1} \in C^1(\mathcal{B}; \mathcal{A})$.

$$\phi(u) = \operatorname*{stat}_{v \in \mathcal{B}} [a(v) + \langle v, u \rangle] \qquad \forall\, u \in \mathcal{A},$$
$$a(v) = \operatorname*{stat}_{u \in \mathcal{A}} [\phi(u) - \langle v, u \rangle] \qquad \forall\, v \in \mathcal{B}.$$

- Example: $\mathcal{A} = \mathcal{B} = \boldsymbol{R}^n \setminus \{0\}$.

$$\phi(u) = 1/|u|, \qquad a(v) = 2|v|^{1/2}.$$

- **Stat-quad duality:** $\mathcal{A}, \hat{\mathcal{B}}$ open; $C \in \mathcal{L}(\mathcal{U}; \mathcal{U})$, symmetric and invertible; $\eta^{-1} \in C^1(\hat{\mathcal{B}}; \mathcal{A})$ with $\eta(u) \doteq D\phi(u) - Cu$.

$$\phi(u) = \operatorname*{stat}_{v \in \mathcal{B}} \left[a(v) + \tfrac{1}{2}\langle v - u, C(v - u)\rangle\right] \qquad \forall\, u \in \mathcal{A},$$
$$a(v) = \operatorname*{stat}_{u \in \mathcal{A}} \left[\phi(u) - \tfrac{1}{2}\langle v - u, C(v - u)\rangle\right] \qquad \forall\, v \in \mathcal{B}.$$

- Example: $\mathcal{A}, \hat{\mathcal{B}} = \boldsymbol{R}^n$; $P, C, P - C$ symmetric, nonsingular.

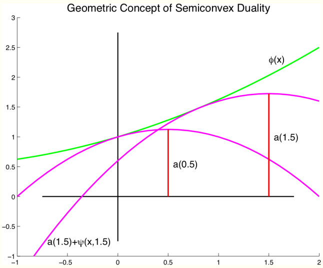
$$\phi(u) = \tfrac{1}{2} u' P u, \qquad a(v) = \tfrac{1}{2} v' C (C - P)^{-1} P v.$$

Mass-Spring Example

- Using stationary action to obtain a differential Riccati equation, the solution is

$$P(t) = R(t) = \frac{-\cot(\omega t)}{\gamma}, \quad Q(t) = \frac{\operatorname{cosec}(\omega t)}{\gamma}.$$

- Naive use of closed-form solution past asymptotes yields correct stationary action, and solution to TPBVP.
- Propagation aided via stat-quad duality:



- Stat-quad dual of quadratics corresponding to $P(t)$, $Q(t)$, $R(t)$, obtained from following (with using duality matrix C):

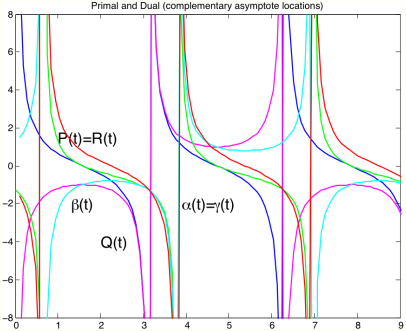
$$\alpha(t) = C - C[C + P(t)]^{-1} C,$$
$$\beta(t) = C[C + P_(t)]^{-1} Q(t),$$
$$\gamma(t) = R(t) - Q^T(t)[C + P(t)]^{-1} Q(t).$$

Propagation Through Asymptotes

- Propagation through stat-quad duality:

- Stat-dual satisfies:

$$\begin{aligned}\dot{\alpha}(t) &= -\alpha(t)[D^{-1} + C^{-1}BC^{-1}]\alpha(t),\\ \dot{\beta}(t) &= -\alpha(t)[D^{-1} + C^{-1}BC^{-1}]\beta(t)\\ &\quad + BC^{-1}\beta(t),\\ \dot{\gamma}(t) &= -\beta^T(t)[D^{-1} + C^{-1}BC^{-1}]\beta(t).\end{aligned}$$



- Locations of asymptotes may be different between primal and dual.
- Propagation recipe:

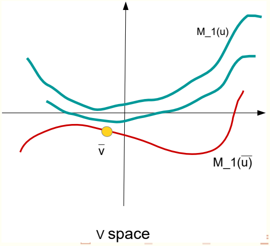
  1. Propagate primal [dual] Riccati until approaching asymptote.
  2. Switch to dual [primal] Riccati until approaching dual [primal] asymptote, and return to step 1.

The Theory of Iterated Staticization

- **When is**

$$\operatorname*{stat}_{u\in\mathcal{U}}\operatorname*{stat}_{\alpha\in\mathcal{A}} F(u,\alpha) = \operatorname*{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} F(u,\alpha) = \operatorname*{stat}_{\alpha\in\mathcal{A}}\operatorname*{stat}_{u\in\mathcal{U}} F(u,\alpha) \textbf{ ?}$$

- This is a surprisingly deep question.
    - (Definitely **not** $\frac{d}{du}\frac{dF}{d\alpha} = \frac{d^2F}{du\,d\alpha} = \frac{d}{d\alpha}\frac{dF}{du}$!)
- Counterexample on $\mathcal{U} = \mathcal{A} = \boldsymbol{R}$: $F(u,\alpha) = u(\alpha^2 - 1)$.
    - $\operatorname{stat}_{\alpha\in\mathcal{A}}\operatorname{stat}_{u\in\mathcal{U}} F(u,\alpha) = 0 = \operatorname{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} F(u,\alpha)$.
    - $\operatorname{stat}_{u\in\mathcal{U}}\operatorname{stat}_{\alpha\in\mathcal{A}} F(u,\alpha)$ does not exist.
- Letting $\mathcal{M}_1(u) \doteq \operatorname{argstat}_{v\in\mathcal{V}} F(u,v)$, the underlying condition is that $d(\bar v, \mathcal{M}_1(u))$ grow at most at a Lipschitz rate in neighborhood of $(\bar u, \bar v) \in \operatorname{argstat}_{(u,v)} F(u,v)$.



Iterated Staticization Problem

- The semi-quadratic case:

$$F(u,\alpha) \doteq f_1(\alpha) + \langle f_2(\alpha), u\rangle_{\mathcal{U}} + \tfrac{1}{2}\langle \bar{B}_3(\alpha)u, u\rangle_{\mathcal{U}}.$$

  - Boundedness condition on Moore-Penrose pseudo-inverse, $\bar{B}_3^{\#}(\alpha)$, and additional technical conditions.

- Then, if the former exists,

$$\operatorname*{stat}_{\alpha\in\mathcal{A}} \operatorname*{stat}_{u\in\mathcal{U}} F(u,\alpha) = \operatorname*{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} F(u,\alpha).$$

- The uniformly locally Morse case:
- Very roughly: F is Morse if $F_\alpha(\hat{u},\hat{\alpha}) = 0$ implies $F_{\alpha\alpha}(\hat{u},\hat{\alpha})$ is invertible.
- Then, if the former exists,

$$\operatorname*{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} F(u,\alpha) = \operatorname*{stat}_{\alpha\in\mathcal{A}} \operatorname*{stat}_{u\in\mathcal{U}} F(u,\alpha).$$

Gravitational N-Body Problem (Motivation for the Above)

- Dynamics: $\dot{\xi} = u,\ \ \xi(0) = x = (x^1, x^2, \dots x^N),\ \ u \in \mathcal{U} = L_2^{loc}$.
- Action functional (with all masses set to 1):

$$\overline{J}^{\infty}(t,x,u;z) = \int_0^t \tfrac{1}{2}|u(r)|^2 + \hat{G} \operatorname*{stat}_{\alpha \in \mathcal{A}_0} \sum_{i,j} \left[\alpha^{i,j} - \tfrac{(\alpha^{i,j})^3 |\xi^i(r) - \xi^j(r)|^2}{2}\right] dr + \psi^{\infty}(\xi(t),z)$$

$$= \operatorname*{stat}_{\alpha(\cdot) \in \mathcal{A}} \left\{ \int_0^t \tfrac{1}{2}|u(r)|^2 + \hat{G} \sum_{i,j} \left[\alpha^{i,j} - \tfrac{(\alpha^{i,j})^3 |\xi^i(r) - \xi^j(r)|^2}{2}\right] dr + \psi^{\infty}(\xi(t),z) \right\}$$

($\mathcal{A}$ – measurable α components in $(0,\infty)$)

- Value function:

$$\overline{W}^{\infty}(t,x;z) = \operatorname*{stat}_{u \in \mathcal{U}} \operatorname*{stat}_{\alpha(\cdot) \in \mathcal{A}} \left\{ \int_0^t \tfrac{1}{2}|u(r)|^2 + \hat{G} \sum_{i,j} \left[\alpha^{i,j} - \tfrac{(\alpha^{i,j})^3 |\xi^i(r) - \xi^j(r)|^2}{2}\right] dr + \psi^{\infty}(\xi(t),z) \right\}.$$

The N-Body Problem (Motivation)

- $J^\infty(t,x,u,\alpha^*;z)$ is semi-quadratic in u.
- $J^\infty(t,x,u,\alpha;z)$ is locally uniformly Morse in α.
- Hence

$$\begin{aligned}\overline{W}^\infty(t,x;z) &= \operatorname*{stat}_{u\in\mathcal{U}}\ \operatorname*{stat}_{\alpha(\cdot)\in\mathcal{A}} \left\{ \int_0^t \frac{1}{2}|u(r)|^2 + \hat{G}\sum_{i,j}\left[\alpha^{i,j} - \frac{(\alpha^{i,j})^3|\xi^i(r)-\xi^j(r)|^2}{2}\right] dr \right.\\ &\qquad\qquad \left. + \psi^\infty(\xi(t),z)\right\}.\\ &= \operatorname*{stat}_{u\in\mathcal{U}}\ \operatorname*{stat}_{\alpha(\cdot)\in\mathcal{A}} J^\infty(t,x,u,\alpha;z)\\ &= \operatorname*{stat}_{\alpha(\cdot)\in\mathcal{A}}\ \operatorname*{stat}_{u\in\mathcal{U}} J^\infty(t,x,u,\alpha;z)\\ &= \operatorname*{stat}_{\alpha\in\mathcal{A}} \mathcal{W}^{\alpha,\infty}(t,x;z).\end{aligned}$$

- For each $\alpha\in\mathcal{A}$, $\mathcal{W}^{\alpha,\infty}(t,x;z)$ is solution of an LQ control problem.

The N-Body Fundamental Solution as a Set (Motivation)

- We have

$$\mathcal{W}^{\alpha,\infty}(t,x;z) = \frac{1}{2}\left[x^T P_t^\infty(\alpha)x + 2z^T Q_t^\infty(\alpha)x + z^T R_t^\infty(\alpha)z + r_t^\infty(\alpha)\right]$$

  where $P_t^\infty, Q_t^\infty, R_t^\infty$ are solutions of Riccati equations and r_t^∞ is an integral.

- The game value function is:

$$\begin{aligned}\overline{W}^\infty(t,x;z) &= \operatorname*{stat}_{\alpha\in\mathcal{A}} \frac{1}{2}\left[x^T P_t^\infty(\alpha)x + 2z^T Q_t^\infty(\alpha)x + z^T R_t^\infty(\alpha)z + r_t^\infty(\alpha)\right] \\ &= \operatorname*{stat}_{(P,Q,R,r)\in\mathcal{G}_t} \frac{1}{2}\left[x^T P x + 2z^T Q x + z^T R z + r\right].\end{aligned}$$

- The set

$$\mathcal{G}_t \doteq \{P_t^\infty(\alpha), Q_t^\infty(\alpha), R_t^\infty(\alpha), r_t^\infty(\alpha) \,|\, \alpha \in \mathcal{A}\}$$

  represents the fundamental solution of n-body TPBVPs.

Schrödinger Similarity

- The Schrödinger equation case is similar.

$$\bar{J}(s,x,u,\alpha) \doteq \mathsf{E}\left\{ \int_s^t \frac{m}{2} u_r^T u_r - V(\xi_r)\,dr + \phi(\xi_t) \right\},$$

where

$$d\xi_r = u_r\,dr + \sigma \tfrac{1+i}{\sqrt{2}}\,dB_r.$$

- The stat operations will be over complex-valued, stochastic processes.

Part 2:
Converting the Second-Order HJ PDE Problem into a First-Order HJ PDE Problem and Associated Ramifications

Stochastic Control Problem

- SDE dynamics:

$$d\xi_t = f(\xi_t, u_t)\,dt + \mu\,dB_t, \qquad \xi_s = x \in \boldsymbol{R}^n.$$

- Payoff:

$$\mathcal{J}(s,x,u) \doteq \mathsf{E}\Big\{ \int_s^T L(\xi_t,u_t)\,dt + \Psi(\xi_T) \Big\}.$$

- Can use a stat-quad duality representation for a variety of terminal costs:

$$\Psi(x) \doteq \operatorname*{stat}_{z\in\boldsymbol{R}^n} \left\{ \hat\gamma(z) + \tfrac{1}{2}(x-z)^T \bar M (x-z) \right\} \doteq \operatorname*{stat}_{z\in\boldsymbol{R}^n} \{\psi(x;z)\},$$

$$\hat\gamma(z) \doteq \operatorname*{stat}_{x\in\boldsymbol{R}^n} \left\{ \Psi(x) - \tfrac{1}{2}(x-z)^T \bar M (x-z) \right\},$$

- Then,

$$\mathcal{J}(s,x,u) \doteq \operatorname*{stat}_{\zeta\in\mathcal{Z}} \{ J(s,x,u;z) \},$$

$$J(s,x,u;z) \doteq \mathsf{E}\Big\{ \int_s^T L(\xi_t,u_t)\,dt + \psi(\xi_T;z) \Big\}.$$

- We will henceforth focus on $J(s,x,u;z)$.

Dynamic Programming and Stat-Quad Duality

- Value function:
$$W(s,x;z) = \operatorname*{stat}_{u\in\mathcal{U}_s} J(s,x,u;z).$$
- Making the standard assumptions for existence of solution of HJ PDE, and verification theorem for traditional optimization (plus a bit more if the staticization is not optimization).
- Associated HJ PDE problem (with $A \doteq \sigma\sigma^T$):
$$\begin{aligned} 0 &= W_t + \operatorname*{stat}_{v\in U}\left\{ f(x,v)^T W_x + L(x,v)\right\} + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] \\ &\doteq W_t + H_0(x,W_x) + \mathcal{Q}_0(x,W_x) + \tfrac{1}{2}\operatorname{tr}[AW_{xx}], \\ W(T,x;z) &= \psi(x;z). \end{aligned}$$

  $\mathcal{Q}_0$ is a quadratic function; putting all the non-linear/quadratic terms in H_0.)
- Stat-quad duality (with $\mathcal{Q}(x,p,\alpha,\beta) \doteq \frac{c_1}{2}|x-\alpha|^2 + \frac{c_2}{2}|p-\beta|^2$ and $|c_1|, |c_2|$ sufficiently large):
$$\begin{aligned} H_0(x,p) &= \operatorname*{stat}_{(\alpha,\beta)\in\mathbf{R}^{2n}}\left[G_0(\alpha,\beta) + \mathcal{Q}(x,p,\alpha,\beta)\right], \\ G_0(\alpha,\beta) &= \operatorname*{stat}_{(x,p)\in\mathbf{R}^{2n}}\left[H_0(x,p) - \mathcal{Q}(x,p,\alpha,\beta)\right]. \end{aligned}$$

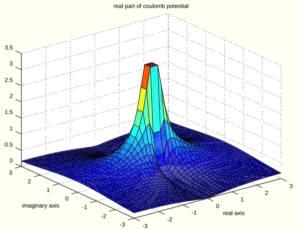
Recall Examples

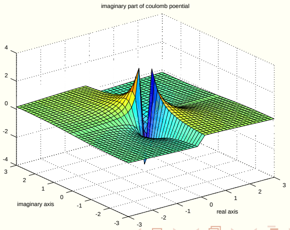
- Additive inverse of the gravitational potential (with $\widehat{G} \doteq (3/2)^{3/2} G$).

$$-V(x) = \frac{\widehat{G} m_0 m}{|x|} = \widehat{G} m_0 m \operatorname*{stat}_{\alpha \in [0,\infty)} \left\{ \alpha - \frac{\alpha^3 |x|^2}{2} \right\}.$$

- Extension of the Coulomb potential to $\mathbb{C}^3$:

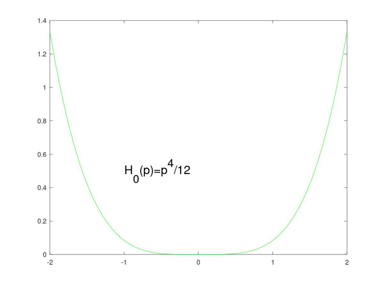
$$-V(x) = \frac{\mu_c}{\sqrt{x^T x}} = \hat{\mu} \operatorname*{stat}_{\alpha \in \mathcal{H}^+} \left[\alpha - \frac{\alpha^3 (x^T x)}{2} \right].$$

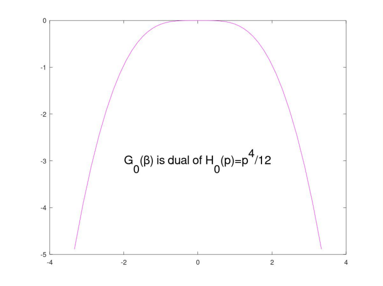




Another Example

- These examples include the stat-quad duality in the gradient variable as well.
- Simple example where $H_0(x,p) = \frac{p^4}{12}$.





Dynamic Programming and Stat-Quad Duality

- Revised HJ PDE problem (with $A \doteq \sigma\sigma^T$):

$$0 = W_t + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] + \operatorname*{stat}_{(\alpha,\beta)\in \mathbb{R}^{2n}} \left\{ G_0(\alpha,\beta) + \mathcal{Q}(x, W_x, \alpha, \beta) + \mathcal{Q}_0(x, W_x) \right\},$$

$$W(T, x; z) = \psi(x; z).$$

- Note that aside from the staticization over the newly introduced parameters α, β, the Hamiltonian is quadratic.
- The HJ PDE is

$$0 = W_t + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] + \operatorname*{stat}_{(\alpha,\beta)\in \mathbb{R}^{2n}} \left\{ G_0(\alpha,\beta) + \frac{c_1}{2}|\alpha|^2 + \frac{c_2}{2}|\beta|^2 + k_1\alpha^T x + k_2\beta^T W_x + \mathcal{Q}_1(x, W_x) \right\},$$

where $\mathcal{Q}_1(x, W_x)$ is quadratic.

HJ PDE and Iterated Staticization

- Use a stat-quad dual of the quadratic, $\mathcal{Q}_1$ to get it in a control form, yielding

$$\begin{aligned}0 = {} & W_t + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] \\ & + \operatorname*{stat}_{(\alpha,\beta)\in\boldsymbol{R}^{2n}} \Big\{ G_0(\alpha,\beta) + \frac{c_1}{2}|\alpha|^2 + \frac{c_2}{2}|\beta|^2 + k_1\alpha^T x + k_2\beta^T W_x \\ & + \operatorname*{stat}_{w\in\boldsymbol{R}^n} \big[(B_1 w + B_2)^T W_x + \tfrac{1}{2}w^T\Gamma_1 w + \tfrac{1}{2}x^T\Gamma_2 x + B_3^T x + k_3\big]\Big\}.\end{aligned}$$

- Using one of the iterated staticization results, this is

$$\begin{aligned}0 = {} & W_t + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] \\ & + \operatorname*{stat}_{(\alpha,\beta,w)\in\boldsymbol{R}^{3n}} \Big\{ G_0(\alpha,\beta) + \frac{c_1}{2}|\alpha|^2 + \frac{c_2}{2}|\beta|^2 + k_1\alpha^T x + k_2\beta^T W_x \\ & + (B_1 w + B_2)^T W_x + \tfrac{1}{2}w^T\Gamma_1 w + \tfrac{1}{2}x^T\Gamma_2 x + B_3^T x + k_3\Big\}.\end{aligned}$$

Dynamic Programming and Iterated Staticization

- The associated control problem is

$$
\begin{aligned}
&d\xi_t = (k_2\beta_t + B_1 w_t + B_2)\,dt + \sigma\,dB_t, \qquad \xi_s = x,\\
&J^f(s,x,w,\alpha,\beta;z) = \int_s^T L^f(\xi_t,w_t,\alpha_t,\beta_t)\,dt + \psi(\xi_t;z),\\
&W^f(s,x;z) = \operatorname*{stat}_{(\alpha_\cdot,\beta_\cdot,w_\cdot)\in\bar{\mathcal{O}}_s\times\mathcal{W}_s} J^f(s,x,w,\alpha,\beta;z),\\
&L^f(x,w,\alpha,\beta) \doteq G_0(\alpha,\beta) + \frac{c_1}{2}|\alpha|^2 + \frac{c_2}{2}|\beta|^2\\
&\qquad\qquad + \tfrac{1}{2}w^T\Gamma_1 w + \tfrac{1}{2}x^T\Gamma_2 x + (k_1\alpha + B_3)^T x + k_3.
\end{aligned}
$$

- $\bar{\mathcal{O}}_s$ is a space of stochastic, adapted, right-continuous, square-integrable controls.
- Using iterated staticization again (now over infinite-dimensional spaces),

$$
\begin{aligned}
W^f(s,x;z) &= \operatorname*{stat}_{(\alpha_\cdot,\beta_\cdot)\in\mathcal{O}_s}\ \operatorname*{stat}_{w_\cdot\in\mathcal{W}_s} J^f(s,x,w,\alpha,\beta;z)\\
&\doteq \operatorname*{stat}_{(\alpha_\cdot,\beta_\cdot)\in\mathcal{O}_s} W^{\alpha_\cdot,\beta_\cdot}(s,x;z).
\end{aligned}
$$

Dynamic Programming and Iterated Staticization

- The HJ PDE associated to value function $W^{\alpha.,\beta.}$ is

$$\begin{aligned}0 = {} & W_t + \tfrac{1}{2}\operatorname{tr}[AW_{xx}] \\ & + G_0(\alpha_t, \beta_t) + \frac{c_1}{2}|\alpha_t|^2 + \frac{c_2}{2}|\beta_t|^2 + (k_1\alpha_t + B_3)^T x + \tfrac{1}{2}x^T\Gamma_2 x + k_3 \\ & + (k_2\beta_t + B_2)^T W_x - \tfrac{1}{2}W_x^T\Gamma_3 W_x, \\ & W(T, x; z) = \psi(x; z).\end{aligned}$$

- This is a linear-quadratic problem, indexed by $\alpha., \beta.$.
- The solution has the form

$$W^{\alpha.,\beta.} = \tfrac{1}{2}\binom{x}{z}^T \Pi_t \binom{x}{z} + \pi_t^T \binom{x}{z} + \hat{\gamma}_t(z),$$

  where $\Pi.$ satisfies a differential Riccati equation, and $\pi., \gamma.$ satisfy ODEs with appropriate initial data.

- That was a key step!

Fundamental Reformulation

- **The value function $W^{\alpha.,\beta.}$ is generated by deterministic, fundamental control problem (noting suppressed initial data).**
- The dynamics are differential Riccati equation (DRE) and linear ODEs

$$\dot{\Pi}_t = \bar{F}_1(\Pi_t), \quad \dot{\pi}_t = \bar{F}_2(\Pi_t, \pi_t, \alpha_t, \beta_t), \quad \dot{\gamma}_t = \bar{F}_3(\Pi_t, \pi_t, \alpha_t, \beta_t).$$

- The initial conditions are

$$\Pi_s = \bar{\Pi} \doteq \begin{bmatrix} M & -M \\ -M & M \end{bmatrix}, \quad \pi_s = \bar{\pi} = (0,0)^T, \quad \gamma_s = \hat{\gamma}(z).$$

- The (terminal-cost) payoff and value function are

$$\begin{aligned} &\bar{J}(s, \bar{\Pi}, \bar{\pi}, \bar{\gamma}; x, z) = \tfrac{1}{2} \binom{x}{z}^T \Pi_T \binom{x}{z} + \pi_T^T \binom{x}{z} + \gamma_T(z), \\ &W^f(s, \bar{\Pi}, \bar{\pi}, \bar{\gamma}; x, z) = W^f(s, x, z) \\ &= \operatorname*{stat}_{(\alpha.,\beta.)\in\mathcal{O}_s} W^{\alpha.,\beta.}(s, x; z) = \operatorname*{stat}_{(\alpha.,\beta.)\in\mathcal{O}_s} \bar{J}(s, \bar{\Pi}, \bar{\pi}, \bar{\gamma}; x, z). \end{aligned}$$

- $\mathcal{O}_s = L_2$. x **is now a parameter.**

Final Fundamental Reformulation

- Note that $\frac{1}{2}\binom{x}{z}^T \Pi_T \binom{x}{z}$ is an additive, uncontrolled term.
- Let

$$\breve{W}(t,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) \doteq W^f(t,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) - \frac{1}{2}\binom{x}{z}^T \Pi_T \binom{x}{z},$$
$$\breve{\psi}(\pi,\gamma;x,z) \doteq \pi^T\binom{x}{z} + \gamma.$$

- **$\breve{W}$ is the value function of the deterministic, terminal-cost, fundamental control problem given by**

$$\begin{aligned}\breve{W}(s,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) &= \operatorname*{stat}_{(\alpha.,\beta.)\in\mathcal{O}_s}\left\{\breve{\psi}(\pi_T(\alpha.,\beta.),\gamma(\alpha.,\beta.);x,z)\right\}\\ &= \operatorname*{stat}_{(\alpha.,\beta.)\in\mathcal{O}_s}\left\{\breve{J}(s,\bar{\Pi},\bar{\pi},\bar{\gamma},\alpha,\beta;x;z).\right\}.\end{aligned}$$

- The dynamics are linear ODEs

$$\dot{\pi}_t = \bar{F}_2(\Pi_t,\pi_t,\alpha_t,\beta_t), \quad \dot{\gamma}_t = \bar{F}_3(\Pi_t,\pi_t,\alpha_t,\beta_t).$$

- The initial conditions are: $\pi_s = \bar{\pi} = (0,0)^T, \quad \gamma_s = \hat{\gamma}(z).$
- **The stochastic control problem has been converted to a deterministic control problem.**

Newly Available Approaches (Max-Plus)

- The above formulation as a terminal-cost deterministic problem has an associated HJ PDE

$$0 = \breve{W}_t + \operatorname*{stat}_{(\alpha,\beta)\in\mathbf{R}^{2n}} \left\{ \bar{F}_2(\Pi,\pi,\alpha,\beta)\cdot\breve{W}_\pi + \bar{F}_3(\Pi,\pi,\alpha,\beta)\breve{W}_\gamma \right\}$$

$$\breve{W}(T,\Pi,\pi,\gamma;x,z) = \pi^T \binom{x}{z} + \gamma.$$

- This is a first-order HJ PDE over $n+1$ dimensional state space (π,γ).
- $\bar{F}_2, \bar{F}_3$ are linear in π, indepedent of γ, quadratic in x, z.
- Low complexity; well below the quantum-spin example.
- Appopriate for max-plus curse-of-dimensionality-free methods.
- Recall that

$$W^f(s,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) = \breve{W}(s,\bar{\Pi},\bar{\pi},\bar{\gamma};x,z) + +\frac{1}{2}\binom{x}{z}^T \Pi_T \binom{x}{z}.$$

Newly Available Approaches (Fundamental Solutions)

- Recall payoff

$$\breve{J}(s, \bar{\Pi}, \bar{\pi}, \bar{\gamma}, \alpha, \beta; x; z) \doteq \breve{\psi}(\pi_T(\alpha., \beta.), \gamma(\alpha., \beta.); x, z).$$

- Differentiate $\breve{J}$ wrt $\alpha., \beta.$ to obtain $\mathrm{argstat}$.
- Obtain an n-dimensional subset of $\mathcal{O}_s$, $\mathcal{G}_s$, that is a fundamental solution set sufficient for computation of solution for any specific $x, z \in \boldsymbol{R}^n$.

Thank you.