Concurrent optimal controller and actuator design for partial differential equations

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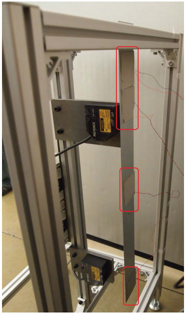
Examples



Linear System

$$\dot{z}(t) = A(r)z(t) + B(r)u(t), \quad t \geq 0; \quad z(0) = z_0$$

- $A(r)$ generates a C_0-semigroup on a Hilbert space $\mathcal{Z}$
- $B(r)$ bounded from $\mathcal{U}$ to $\mathcal{Z}$
- Design actuator location/shape as well as controller
- Design variable $r \in \Omega$ where Ω is compact in some topological space

Common objective: Linear Quadratic (LQ) Control

$$\inf_{u\in L_2(0,\infty;\mathcal{U})} \underbrace{\int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), u(t)\rangle dt}_{J(u,z_0)}$$

Optimal control

If the infinum is finite, then there exists a unique $\Pi \geq 0$ such that for all $z \in D(A)$,

$$\underbrace{(\Pi A + A^*\Pi + C^*C - \Pi BB^*\Pi)z = 0}_{\text{Algebraic Riccati Equation(ARE)}}$$

- Optimal cost $\inf_{u\in L_2(0,\infty;\mathcal{U})} J(u, z_0) = \langle z_0, \Pi z_0\rangle$
- Optimal control $u(t) = -Kz(t)$ where $K = B^*\Pi$

LQ-optimal design

$$\inf_{u \in L_2(0,\infty;\mathcal{U})} \underbrace{\int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), u(t)\rangle dt}_{J_r(u,z_0)}$$

LQ-optimal design

$$\inf_{u \in L_2(0,\infty;\mathcal{U})} \underbrace{\int_0^\infty \left\|z(t)\right\|^2 dt}_{J_r(u,z_0)}$$

$\boxed{\text{C=I, B=0: total energy}}$

LQ-optimal design

$$\inf_{u \in L_2(0,\infty;\mathcal{U})} \underbrace{\int_0^\infty \langle Cz(t), Cz(t)\rangle + \langle u(t), u(t)\rangle dt}_{J_r(u,z_0)}$$

- for each r, optimal cost is $\langle \Pi(r) z_0, z_0\rangle$ where $\Pi(r)$ solves ARE.
- minimize response to the worst $z(0)$

$$\max_{\substack{z_0 \in \mathcal{Z} \\ \|z_0\|=1}} \langle \Pi(r) z_0, z_0\rangle = \|\Pi(r)\|$$

Cost function

$$\hat{\mu} = \inf_{r \in \Omega^m} \|\Pi(r)\|$$

Existence of $\min_{r\in\Omega} \left\|\Pi(r)\right\|$

Theorem 1

Assume that for some $r \in \Omega$

- $\big(A(r), B(r)\big)$ *is stabilizable;*
- $\big(A(r), C\big)$ *is detectable, uniformly in r.*

- Ω *is compact in a topological space*
- *for any sequence $r_n \to r$ in Ω and any $z \in \mathcal{Z}$,*

$$\left\|e^{tA(r_n)}z - e^{tA(r)}z\right\| \to 0$$
$$\left\|e^{tA(r_n)^*}z - e^{tA(r)^*}z\right\| \to 0$$
$$\left\|B(r_n) - B(r)\right\| \to 0.$$

Then, there exists $r^* \in \Omega$ such that

$$\left\|\Pi(r^*)\right\| = \inf_{r\in\Omega} \left\|\Pi(r)\right\|$$

Outline of Proof

- compactness of $\Omega \Rightarrow$ convergent minimizing sequence
- strong convergence of $\Pi(r_n)$ to $\Pi(r^*)$
- Riccati equation satisfied by $\Pi(r^*)$

Generalizes earlier results:

- (Fahroo–Ito 1997): no control operator, exponentially stable second-order systems
- (Morris 2011): A independent of r

Some other related work

- Minimization of H_2, H_∞ cost (Kasinathan–Morris 2014,Morris–Demetriou–Yang 2015)
- Maximization of the decay rate in a string w.r.t. the damping distribution (Cox–Zuazua 1994, Freitas 1998, Cox 1998, Hébrard–Henrot 2003, Münch-Pedregal–Periago 2006. . .)
- Optimization of observability constant: (Privat–Trélat–Zuazua 2013)
- Optimization of minimal time control w.r.t actuator domain, heat equation: (Zheng–Guo–Ali 2015)

Example: Optimal Spatial Distribution of Damping

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + r(x)\frac{\partial w}{\partial t} = 0, \qquad \omega \subset [0,1]$$

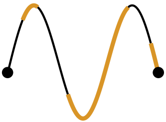
$$w(0,t) = 0, w(1,t) = 0.$$

What is best choice of damping $a(x)$?
Different ways to measure "best"

- decay rate
  - for small mass of damping, constant damping best (Cox & Zuazua)
  - $r(x) = k\chi_\omega(x)$, small k optimum for N modes is at node of $N+1$st and is bad choice (Hebrard & Henrot)
- minimize energy of the system

Example: Vibrating string with viscous damping

Design viscous damping $r(x)$



$$w_{tt} - w_{xx} + r(x)w_t = 0, \quad t > 0,\ 0 < x < 1$$

$$A(r) = \begin{pmatrix} 0 & \mathrm{I} \\ \partial_{xx} & -r(x)\mathrm{I} \end{pmatrix}; \qquad B = 0$$

$$\mathcal{Z} = H_0^1(0,1) \times L^2(0,1)$$

$$\Omega = \left\{ r(x) \in L^\infty(0,1), 0 < r_0 \le r(x) \le r_1, \int_0^1 r(x)dx \le M \right\}$$

compact in $L^\infty(0,1)$ in the weak-star topology

There exists an optimal damping distribution.

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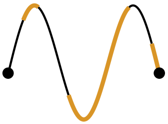

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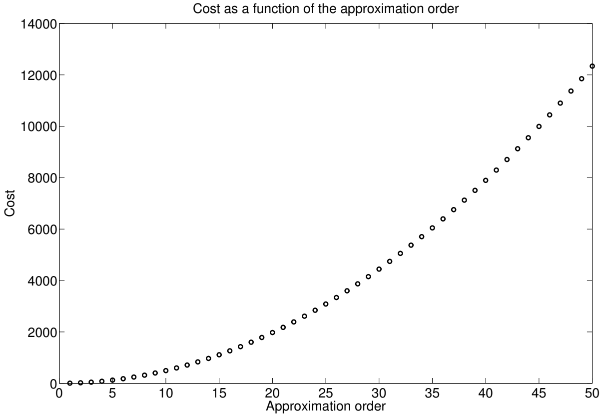
compact in $L^\infty(0,1)$ in the weak-star topology

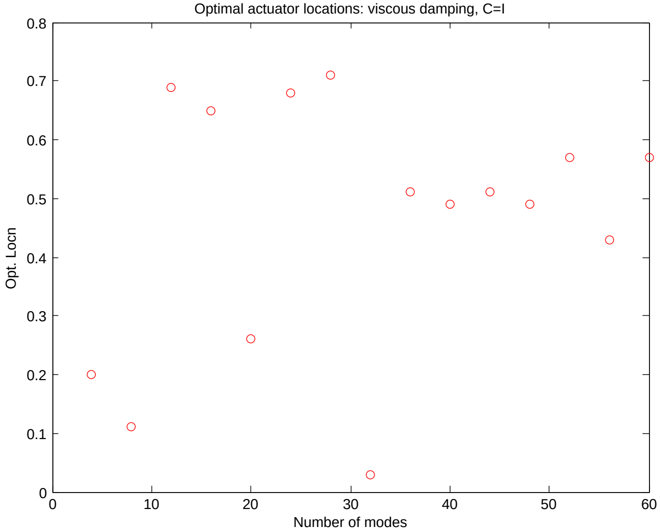
$\boxed{\text{There exists an optimal damping distribution.}}$

Optimal damping; cost is energy (C=I)

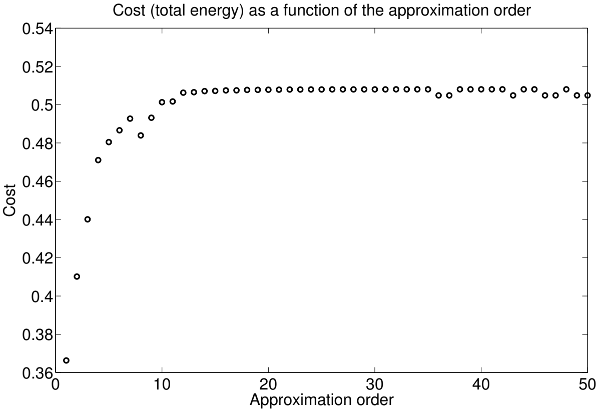
$N =$ number of modes
$r(x) \in \mathrm{Span}\{1, \cos(\pi x), \ldots, \cos((N-1)\pi x)\}$



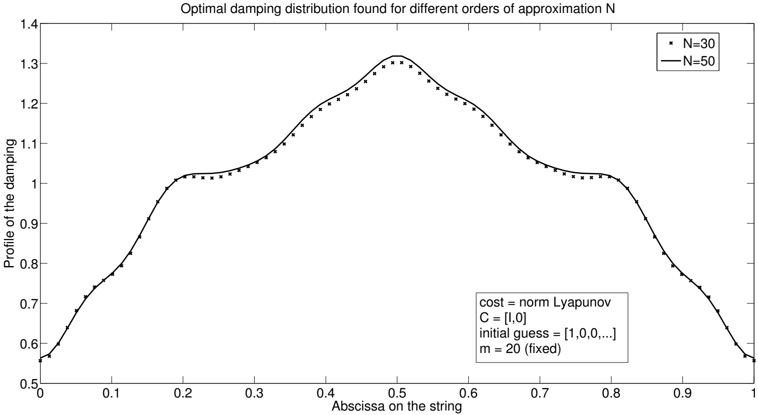
LQ-optimal actuator location ($\|\Pi_n\|$), viscously damped pinned beam $C = I$



Optimal damping; cost with $C = [I\ 0]$



Optimal damping with $C = [I\ 0]$ in cost



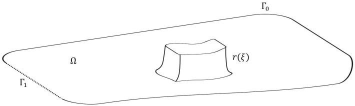
Optimal controller/actuator design: semi-linear PDEs

$$\dot{z}(t) = Az(t) + F(z(t)) + B(\boldsymbol{r})u(t), \quad z(0) = z_0 \in \mathcal{Z}. \qquad \text{(IVP)}$$

- A with domain $\mathcal{D}(A)$ generates a strongly continuous semigroup $T(t)$ on a separable Hilbert space $\mathcal{Z}$.
- $F(\cdot)$ is locally Lipschitz continuous on $\mathcal{Z}$
- input $u(t) \in U_{ad}$ in a Hilbert space $\mathcal{U}$, $U_{ad} = \{u \in L^p(0, T; \mathcal{U}) : \|u\|_p \leq R\}$
- actuator $\boldsymbol{r} \in K_{ad} \subset \mathcal{K}$ in a topological space $\mathcal{K}$
- For each $\boldsymbol{r} \in K_{ad}$, $\mathcal{B}(\boldsymbol{r}) \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, and there exists $M_{\mathcal{B}}$ such that for all $\boldsymbol{r} \in K_{ad}$. $\|\mathcal{B}(\boldsymbol{r})\| \leq M_{\mathcal{B}}$

Example: Nonlinear waves

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}(\xi,t) = \Delta w(\xi,t) + F(w(\xi,t)) + r(\xi)u(t),\\
&w(\xi,0) = w_0(\xi),\ \frac{\partial w}{\partial t}(\xi,0) = v_0(\xi),\quad \xi \in \Omega,\\
&w(\xi,t) = 0, \quad (\xi,t) \in \Gamma_0 \times [0,\infty),\\
&\frac{\partial w}{\partial \nu}(\xi,t) = 0, \quad (\xi,t) \in \Gamma_1 \times [0,\infty).
\end{aligned}
$$



- $F(\zeta) \in C^2(\mathbb{R})$
- There exist $a_0 > 0$ and $b > 1/2$; $|F''(\zeta)| \le a_0(1 + |\zeta|^b)$
- $F(w) = \sin(w)$ in the Sine-Gordon equation;
  $F(w) = |w|^k w,\ k \ge 2$ in the Klein-Gordon equation
- $K_{ad} = \{r \in C^1(\overline{\Omega}) : \|r\|_{C^1} \le 1\} \subset L^2(\Omega)$

Cost Function

$$J(u, \boldsymbol{r}; z_0) = \int_0^T \phi(z(t)) + \psi(u(t))\, dt, \qquad \text{(Cost)}$$

where $\phi(\cdot)$ and $\psi(\cdot)$ are weakly lower semi-continuous positive functionals on $\mathcal{Z}$ and $\mathcal{U}$, respectively. The optimization problem is

$$\begin{cases} \min & J(u, \boldsymbol{r}; z_0) \\ \text{s.t.} & \dot{z}(t) = \mathcal{A}z(t) + \mathcal{F}(z(t)) + \mathcal{B}(\boldsymbol{r})u(t), \quad \text{for all } t \in (0, T] \\ & z(0) = z_0 \\ & u \in U_{ad}, \\ & \boldsymbol{r} \in K_{ad}. \end{cases} \qquad \text{(P)}$$

Existence of an Optimizer

Theorem 1 (Edaletzadeh & Morris, 2018b)

Assume that

- T *is such that the PDE has solution for all admissible* u *and* r.
- $F(\boldsymbol{x})$ *is weakly continuous*
- *Let* K_{ad} *be a convex set, compact in* $\mathbb{K}$. *For all* $\boldsymbol{r}_0 \in K_{ad}$,

$$\lim_{\boldsymbol{r} \to \boldsymbol{r}_0} \|\mathcal{B}(\boldsymbol{r}) - \mathcal{B}(\boldsymbol{r}_0)\|_{\mathcal{L}(\mathcal{U},\mathcal{Z})} = 0.$$

Then there exists a control input $u^o \in U_{ad}$ *together with an actuator location* $\boldsymbol{r}^o \in K_{ad}$, *that solve the optimization problem.*

Outline of Proof

- $J(\boldsymbol{u}, \boldsymbol{r}; \boldsymbol{x}_0)$ is bounded below, and thus it has a finite infimum, say $j(\boldsymbol{x}_0)$. There is a sequence of inputs $\boldsymbol{u}_n \in U_{ad}$ and actuator location $\boldsymbol{r}_n \in K_{ad}$ such that

$$\lim_{n\to\infty} J(\boldsymbol{u}_n, \boldsymbol{r}_n; \boldsymbol{x}_0) \to j(\boldsymbol{x}_0).$$

- U_{ad} is a convex closed bounded subset of $L^p(0, \tau; \mathcal{U})$, $1 < p < \infty$, and so there is a subsequence $\boldsymbol{u}_n \to \boldsymbol{u}^o \in U_{ad}$, weakly, also indicated by $\boldsymbol{u}_n$.
- Compactness of K_{ad} implies that there is a subsequence of $\boldsymbol{r}_n \to \boldsymbol{r}^o \in K_{ad}$, also indicated by $\boldsymbol{r}_n$.

Outline of Proof (cont.)

- By assumption, $\boldsymbol{r}_n \to \boldsymbol{r}^o$ implies

$$\|\mathcal{B}(\boldsymbol{r}_n) - \mathcal{B}(\boldsymbol{r}_o)\|_{\mathcal{L}(\mathcal{U},\mathcal{Z})} \to 0.$$

- Every continuous linear map is weakly continuous and this can be used to show weak convergence of

$$\int_0^t \mathcal{T}(t-s)B(\boldsymbol{r}_n)\boldsymbol{u}_n(s)ds$$

in $C(0,\tau;\mathcal{Z})$.

- Use weak continuity of $\mathcal{F}$ and existence of mild solution to show convergence of costs.

Characterizing the Optimum

Assumption 1

1. *$F(\cdot)$ is Gateaux differentiable on $\mathcal{Z}$. Indicate its derivative at z by F'_z.*
2. *The mapping $z \mapsto F'_z$ is bounded, i.e., bounded sets in $\mathcal{Z}$ are mapped into bounded sets in $\mathcal{L}(\mathcal{Z})$.*
3. *$B(\boldsymbol{r})$ is Gateaux differentiable with respect to $\boldsymbol{r}$ in $\mathcal{L}(\mathbb{U}, \mathcal{Z})$. Indicate derivative of $B(\boldsymbol{r})$ at $\boldsymbol{r}$ by $B'_{\boldsymbol{r}}$.*
4. *$\mathcal{Z}$, $\mathbb{U}$, $\mathbb{K}$ are Hilbert spaces.*
5. *$J(u, \boldsymbol{r}; z_0) = \int_0^T \langle Qz, z\rangle + \langle Ru, u\rangle \; dt$,
where $Q \in \mathcal{L}(\mathcal{Z}, \mathcal{Z})$, $R \in \mathcal{L}(\mathbb{U}, \mathbb{U})$, $Q \geq 0$, $R > 0$.*

Optimality Conditions

Corollary 2 (Edaletzadeh & Morris, 2018b)

With additional Assumption 1, if $(u^o, \boldsymbol{r}^o)$ *is an interior point of* $U_{ad} \times K_{ad}$*, with optimal trajectory* z^o*, initial condition* z_0*, for cost*

$$J(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}) = \int_0^\tau \langle \mathcal{Q}\boldsymbol{x}(t), \boldsymbol{x}(t)\rangle + \langle \mathcal{R}\boldsymbol{u}(t), \boldsymbol{u}(t)\rangle_{\mathcal{U}}\, dt,$$

if

$$\dot{p}^o(t) = -(A^* + F'^*_{z^o(t)})p^o(t) - Qz^o(t), \quad p^o(T) = 0$$
$$u^o(t) = -R^{-1}B^*(\boldsymbol{r}^o)p^o(t),$$
$$\int_0^T (B'_{\boldsymbol{r}^o}u^o(t))^* p^o(t)\, dt = 0.$$

Generalization

- Non-linear parabolic PDES (Edaletzadeh & Morris 2019)
- includes Kuramoto-Sivashinsky,
- actuator design space a Banach space
- linear PDEs
  - H_2-control (known disturbance) (Morris, Demetriou & Yang 2015)
  - H_∞-control (unknown disturbance) (Kasinathan & Morris 2014)
  - boundary control (work with M. Tucsnak)

Optimality equations for single-input linear system, $R = 1$

If

- the PDE is linear,
- single input: $B(r)u = b(r)u$ with $b(r) \in \mathcal{Z}$ depending on r
- quadratic cost with $Q \geq 0$, $R = 1$,

the optimality equations reduce to $z^o(t)$ solves the PDE, and letting $\Pi(t)$ indicate the solution to the associated differential Riccati equation,

$$u^o(t) = -\langle b(r^o), \Pi(t) z^o(t) \rangle$$

$$\int_0^T \langle b(r^o), \Pi(t) z^o(t) \rangle \, \langle b_r(r^o), \Pi(t) z^o(t) \rangle \, dt = 0. \qquad (\mathrm{o})$$

Numerical Example: Railway Tracks

$$\begin{cases} \rho a \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial \xi^2}\left(EI \frac{\partial^2 w}{\partial \xi^2} + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}\right) + \mu \frac{\partial w}{\partial t} + kw + \alpha w^3 = b(\xi; r) u(t), \\ w(\xi, 0) = w_0(\xi), \quad \frac{\partial w}{\partial t}(\xi, 0) = v_0(\xi), \\ w(0, t) = w(\ell, t) = 0, \\ EI \frac{\partial^2 w}{\partial \xi^2}(0, t) + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}(0, t) = EI \frac{\partial^2 w}{\partial \xi^2}(\ell, t) + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}(\ell, t) = 0. \end{cases}$$

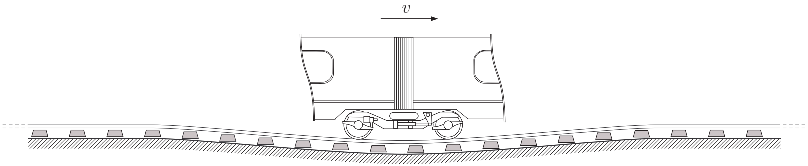


Figure: Force-deflection relationship is nonlinear for railway track beams.

Numerical Example: Railway Tracks

$$
\begin{cases}
\rho a \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial \xi^2}\left(EI \frac{\partial^2 w}{\partial \xi^2} + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}\right) + \mu \frac{\partial w}{\partial t} + kw + \alpha w^3 = b(\xi; r)u(t), \\
w(\xi, 0) = w_0(\xi), \quad \frac{\partial w}{\partial t}(\xi, 0) = v_0(\xi), \\
w(0, t) = w(\ell, t) = 0, \\
EI \frac{\partial^2 w}{\partial \xi^2}(0, t) + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}(0, t) = EI \frac{\partial^2 w}{\partial \xi^2}(\ell, t) + C_d \frac{\partial^3 w}{\partial \xi^2 \partial t}(\ell, t) = 0.
\end{cases}
$$

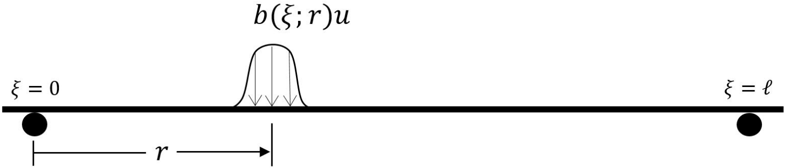


Figure: Schematic of flexible beam

Well-posedness of model

- The dynamics with state $(w, \dot{w})$ are well-posed on

$$\mathcal{Z} = H^2(0,\ell) \cap H_0^1(0,\ell) \times L^2(0,\ell)$$

- nonlinearity $F(w,v) = \begin{bmatrix} 0 \\ -\frac{\alpha}{\rho a} w^3 \end{bmatrix}$ is continuously differentiable and weakly sequentially continuous on $\mathcal{Z}$

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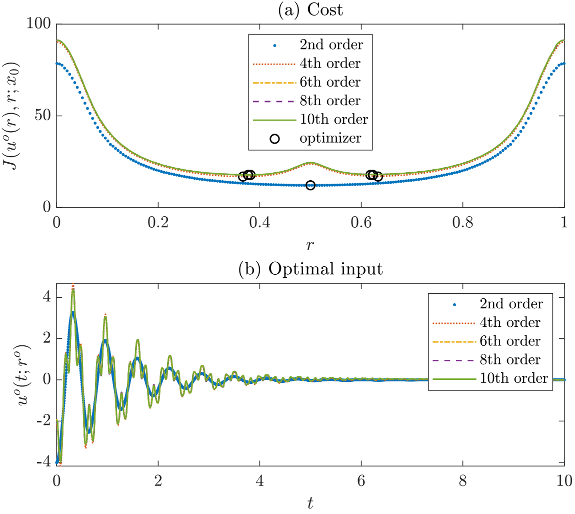
$$\mathcal{Z} = H^2(0,\ell) \cap H_0^1(0,\ell) \times L^2(0,\ell)$$

- nonlinearity $F(w,v) = \begin{bmatrix} 0 \\ -\frac{\alpha}{\rho a} w^3 \end{bmatrix}$ is continuously differentiable and weakly sequentially continuous on $\mathcal{Z}$
- example actuator design problems
  - actuator location: $r \in \mathbb{R}$, $K_{ad} = [0,1] \subset \mathbb{R}$.
  - $b(r) = \chi_r$, r meas. subset of $[0,1]$, $r \in K_{ad}$,

    $K_{ad} := \{\chi_r \in BV(0,1) : Var\{\chi_r(x)\} \leq V,\ |r| = c\} \subset L_1(0,1)$.

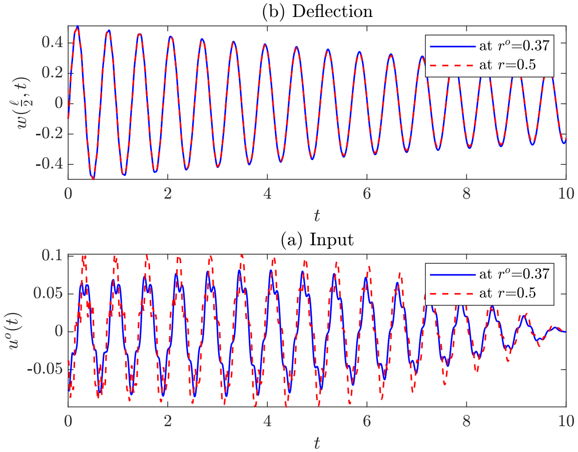
Convergence of Approximate Optimal Control



Parameters	
EI	1
ρa	1
k	1
α	10
μ	0.1
C_d	10^{-4}
ℓ	1
δ	5×10^{-4}
T	10
$\mathcal{Q}$	$\mathcal{I}$
$\mathcal{R}$	$\mathcal{I}$

Figure: Modal approximations (Edalatzadeh)

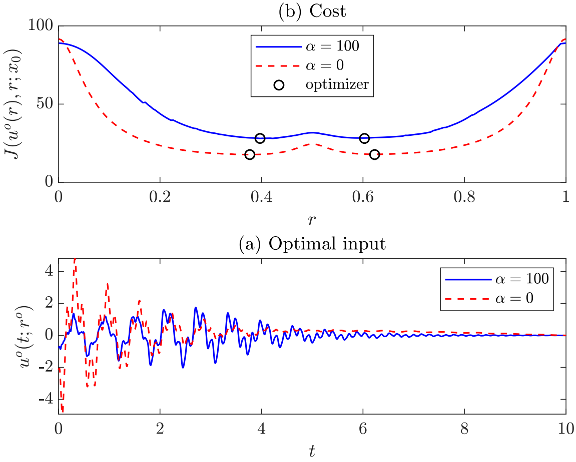
Optimal Control at Optimal Location vs Center Location



Parameters	
EI	1
ρa	1
k	1
μ	0.1
C_d	10^{-4}
α	10
ℓ	1
δ	5×10^{-4}
T	10
$\mathcal{Q}$	$\mathcal{I}$
$\mathcal{R}$	$100 \times \mathcal{I}$

Figure: Comparison of optimal control when the actuator is located at the optimal location and at the center.

Linear vs Nonlinear Control



Parameters	
EI	1
ρa	1
k	1
μ	0.1
C_d	10^{-4}
ℓ	1
δ	5×10^{-4}
T	10
$\mathcal{Q}$	$\mathcal{I}$
$\mathcal{R}$	$\mathcal{I}$

Figure: Cost function and optimal input for linear and nonlinear beam.

Admissible actuator designs

- generally K_{ad} is not naturally a Hilbert space; e.g. $b(r) = \chi_r$, r meas. subset of $[0, 1]$, $r \in K_{ad}$,

$$K_{ad} := \{\chi_r \in BV(0, 1) : Var\{\chi_r(x)\} \leq V, \ |r| = c\} \subset L_1(0, 1).$$

- optimality condition $\int_0^T (\mathcal{B}'_{\boldsymbol{r}^o} \boldsymbol{u}^o(t))^* \boldsymbol{p}^o(t)\, dt = 0$
- possible computational approaches
  - finite-dimensional basis for shape and optimize over coefficients
  - satisfy optimality condition for subset of variations
  - link with topological derivative (Kalise)

Numerical result with linear beam

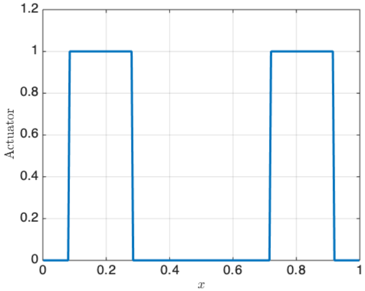
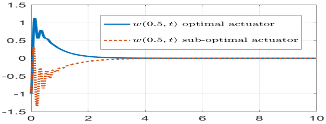
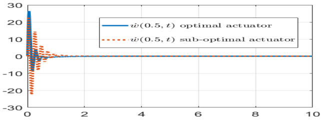


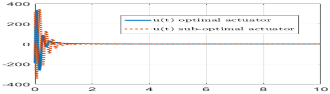
Figure: LQ-optimal actuator. Initial condition $w(x, 0) = \sin(3\pi x)$, $v(x, 0) = 0$, volume constraint of 40% of domain (Kalise)



(a) Controlled displacement



(b) Controlled velocity



(c) Optimal controller

Figure: Closed-loop performance of the optimal actuator against optimal 1-piece actuator $w_s = [0.2, 0.6]$ with same volume.

Another problem with non-parabolic equation

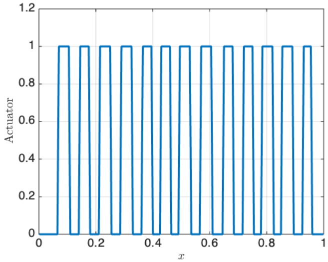


Figure: $N = 40$. With only viscous damping, the optimal actuator splits into multiple components as the number of modes increase.

Summary

- actuator location/choice is part of controller design
- optimal actuator/controller design approach established
- explicit optimality equations
- numerical algorithm for linear systems exists
- no convergence theory for nonlinear PDEs
- shape design numerics not straightforward
- computation for non-parabolic PDEs open problem