Naval Postgraduate School

Computational Optimal Control

Theory & Tools Beyond Nonlinear Programming

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3rd Workshop in Computational Issues in Nonlinear Control
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The views expressed in these slides are not the official views of the US Navy. These views are solely those of the presenter: I. M. Ross, Distinguished Professor & Program Director, Control & Optimization.

What is Optimal Control?

- **Not just LQR, LQG …**
- **Everything is nonlinear == conference theme**
  - Sample Problem:

$$
\overbrace{(B)}^{\textit{problem}}
\left\{
\begin{array}{lrl}
\text{Minimize} & J[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), t_0, t_f] := & \\
& E(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f) + \displaystyle\int_{t_0}^{t_f} F(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt & \Big\} \textit{ cost} \\
\text{Subject to} & \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) & \} \textit{ dynamics} \\
& \boldsymbol{e}^L \leq \boldsymbol{e}(\boldsymbol{x}_0, \boldsymbol{x}_f, t_0, t_f) \leq \boldsymbol{e}^U & \} \textit{ events} \\
& \boldsymbol{h}^L \leq \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \leq \boldsymbol{h}^U & \} \textit{ path}
\end{array}
\right.
$$

Why Optimal Control?

- **Optimal control is the foundational science for:**
  - Nonlinear control
  - Machine learning; deep learning.
    - TOE
  - Economics
  - Pharmacology
  - Physics
  - Management science
  - …

- **Optimal control is also an emerging foundational science for:**
  - Nonlinear programming problems
  - Traveling salesman problems and beyond

Outline: Three Part Theory + One Part Tool

1. **Computational optimal control vs nonlinear programming**

2. **Optimal control programming**
   - Non-introduction introduction

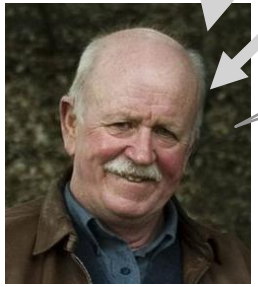
3. **Birkhoff methods for optimal control**
   - Very brief intro to some seemingly magical properties

4. **MATLAB Toolbox for optimal control**
   - How to solve hard problems easily

Theory

Tool

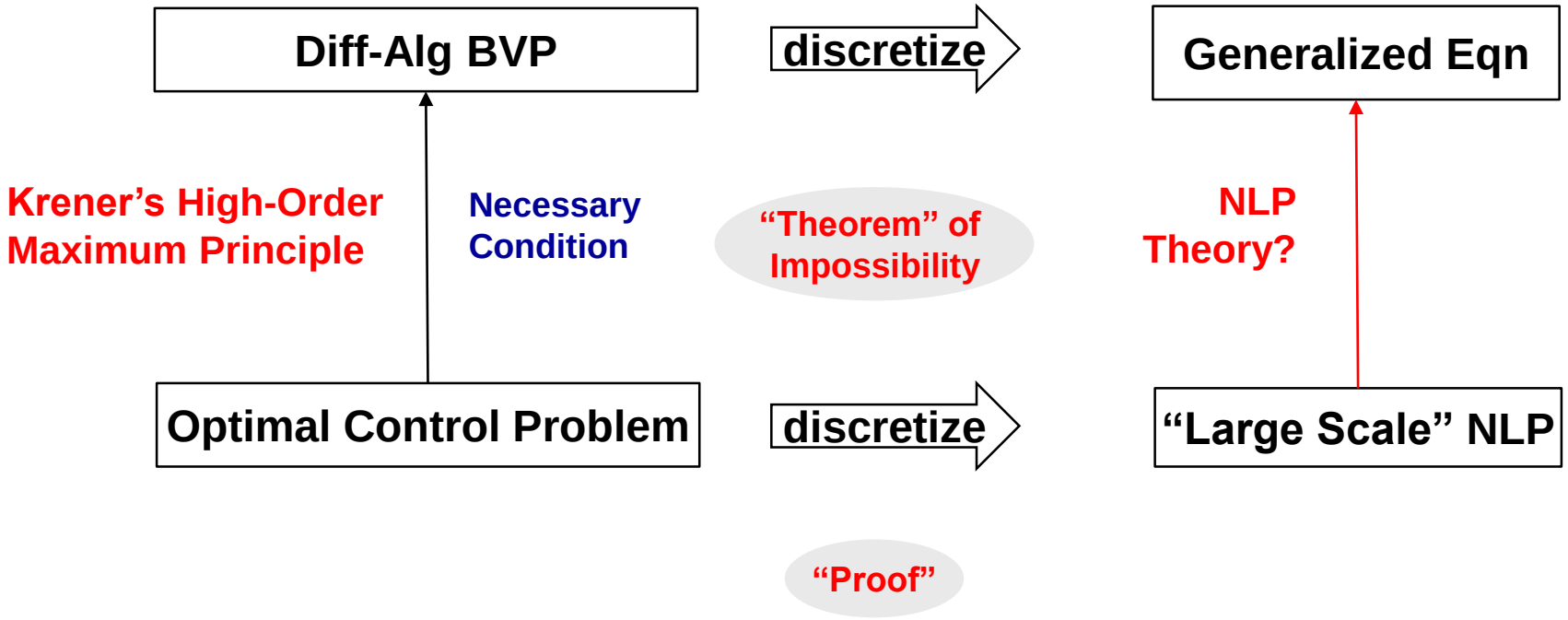
That's the conference theme!



KEY ASSUMPTION

Necessary conditions are actually necessary!

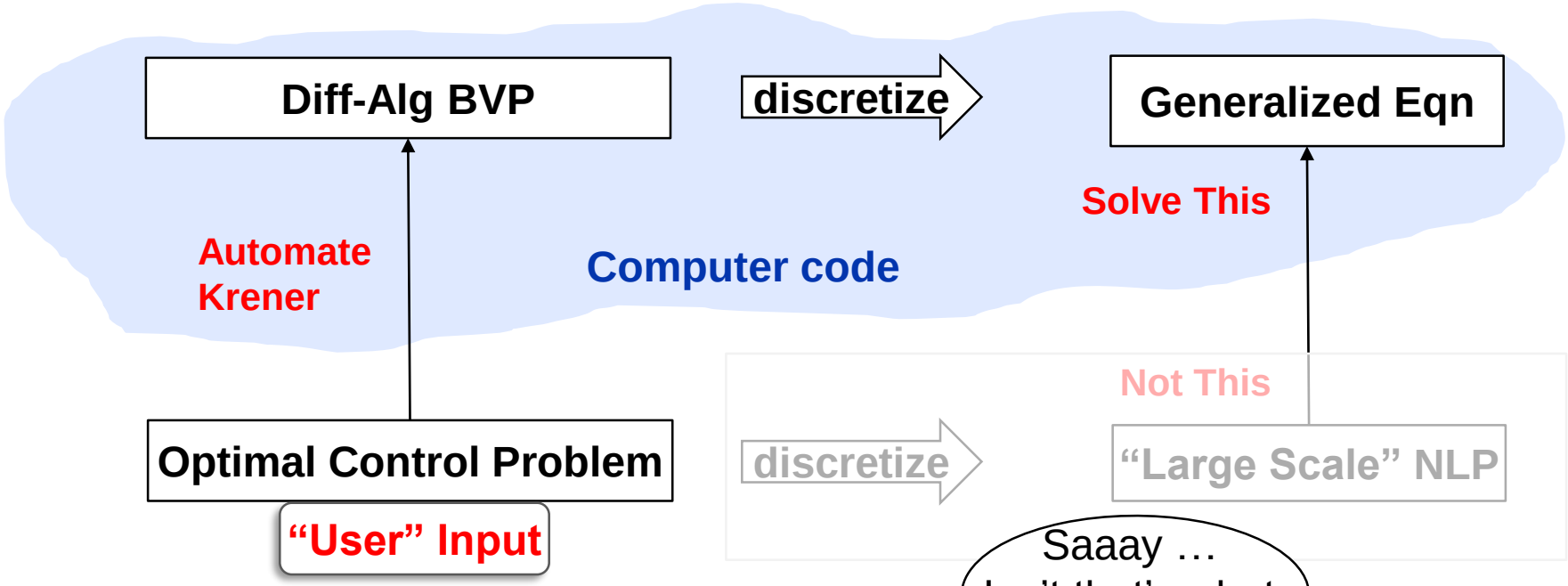
Optimal Control = Large Scale NLP?



- **Optimal control theory has an "infinitely" rich amount of information via:**
  - "Convexity" of time
  - Diffuse measure
  - "Infinite" differentiability wrt time

- **NLP theory can have "at most" countably infinite information.**
  - **Conjecture: Impossible to close the information gap**
  - There are no "Krener-type" conditions in NLP theory
    - Unlikely to discover one

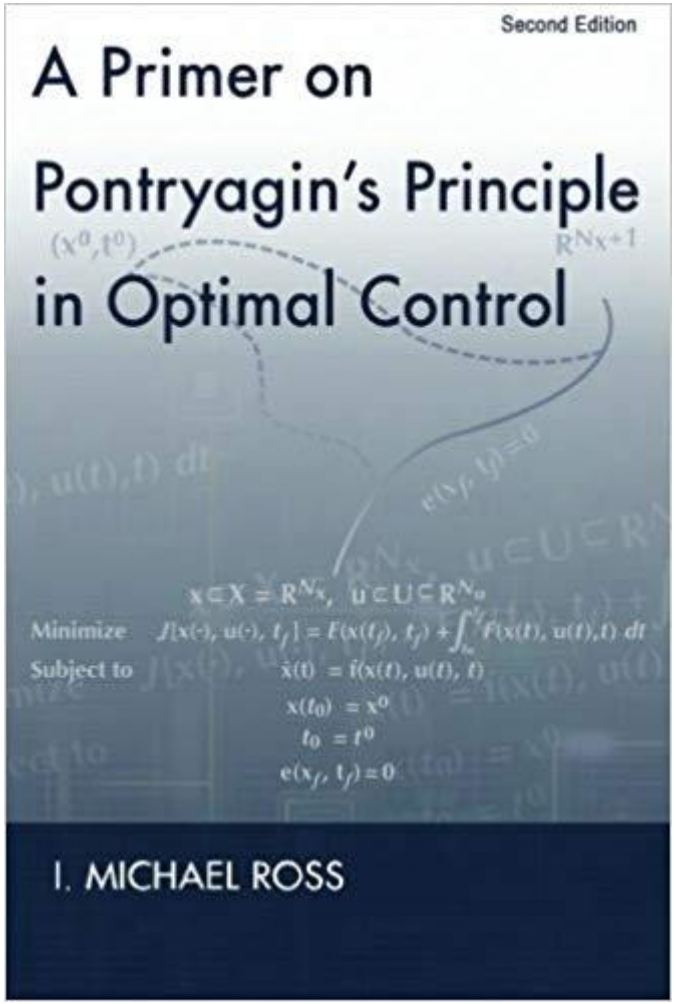
Optimal Control Programming: Big Pic Idea



- **Big Questions:**
  1. Is it possible to automate Krener?
     - No formulas for Krener's higher-order variations

  — Start with automatic Pontryagin; graduate to "low-Fi" K
     - Low-Fi Krener = Generalized Goh-Robbins condition – re "automatic" high order differentiation
  2. How to solve the generalized equation?

Extra, Extra: Read All About It Here!



Yes, Shameless Advertisement

Introduction To A Sacrilegious Idea

- **What is the best way to discretize a nonlinear ODE?**

$$\dot{x} = f(x, u, t)$$

- **What if?**

$$\dot{x} := v$$

Best way to discretize this?!

$$v = f(x, u, t)$$

- **DAE: Differential Algebraic Equation (= horror of horrors?)**

A Quick Pontryagin Math for Sacrilegiousness

$$x \in \mathbb{R}, \quad u \in \mathbb{R}, \quad \tau \in [-1, 1]$$

$$(B) \begin{cases} \textit{Minimize} & J[x(\cdot), u(\cdot)] = E(x(1)) \\ \textit{Subject to} & \dot{x}(\tau) = f(x(\tau), u(\tau)) \\ & x(-1) = 0 \end{cases}$$

$$x \in \mathbb{R}, \quad u \in \mathbb{R}, \quad \tau \in [-1, 1]$$

$$(B^\flat) \begin{cases} \textit{Minimize} & J^\flat[x(\cdot), u(\cdot), v(\cdot)] = E(x(1)) \\ \textit{Subject to} & \dot{x}(\tau) = v(\tau) \\ & f(x(\tau), u(\tau)) - v(t) = 0 \\ & x(-1) = 0 \end{cases}$$

Nonlinear Adjoint Equation ☹

$$
(B^{\flat,\lambda})\left\{
\begin{aligned}
\dot{x} &= v \\
-\dot{\lambda}^{\flat} &= \left[\partial_{\boldsymbol{x}} f(x,u)\right]^T \boldsymbol{\mu}^{\flat} \\
f(x,u) - v &= \mathbf{0} \\
\lambda^{\flat} - \mu^{\flat} &= 0 \\
\left[\partial_u f(x,u)\right]^T \mu^{\flat} &= \mathbf{0} \\
x_0 &= 0 \\
\lambda_f &= \partial_{x_f} E(x_f)
\end{aligned}
\right.
$$

Unnecessary equation

"Create" A New "Flat ♪" Set of Primal-Dual Eqns

$$
(B^{\lambda,\flat})\begin{cases}
\dot{x} - v = 0 \\
\dot{\lambda} - \omega = 0 \\
f(x,u) - v = 0 \\
\partial_x f(x,u)\lambda + \omega = 0 \\
\partial_u f(x,u)\lambda = 0 \\
x(-1) = 0 \\
\lambda(1) - \partial_{x(1)} E(x(1)) = 0
\end{cases}
$$

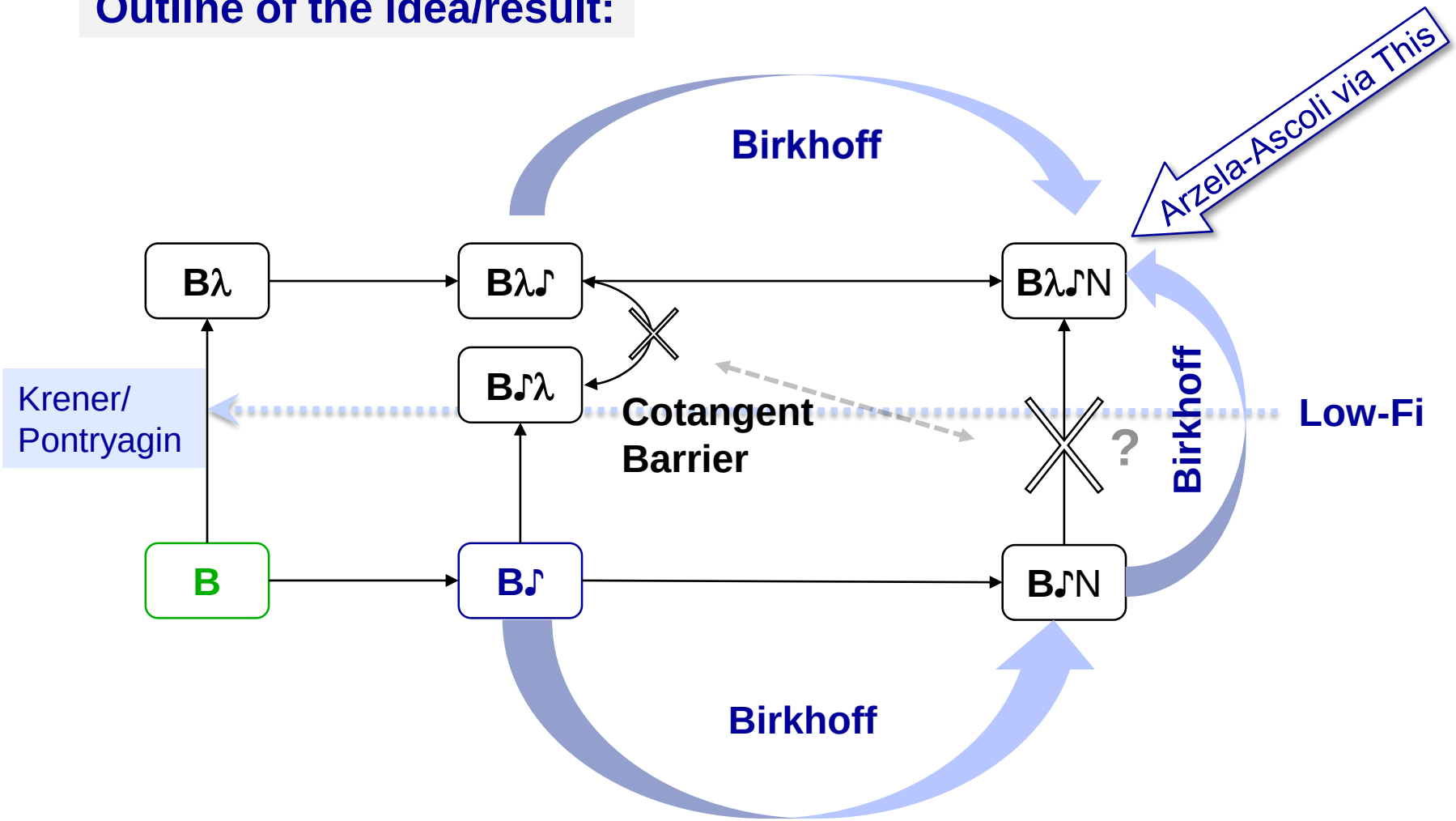
DAE Galore: "Inescapable" DAE

Discretize and Solve this Generalized Eqn via "Arzela-Ascoli" Sequencing

$$B^{\lambda,\flat} \not\equiv B^{\flat,\lambda}$$

Low-Fi Auto-Krener Via Birkhoff

Outline of the idea/result:



Brief Intro to Birkhoff Interpolation

- **Birkhoff generalizes Lagrange/Hermite:**

— Example from Wang, Samson, Zhao, SISC, 36 (3); 2014

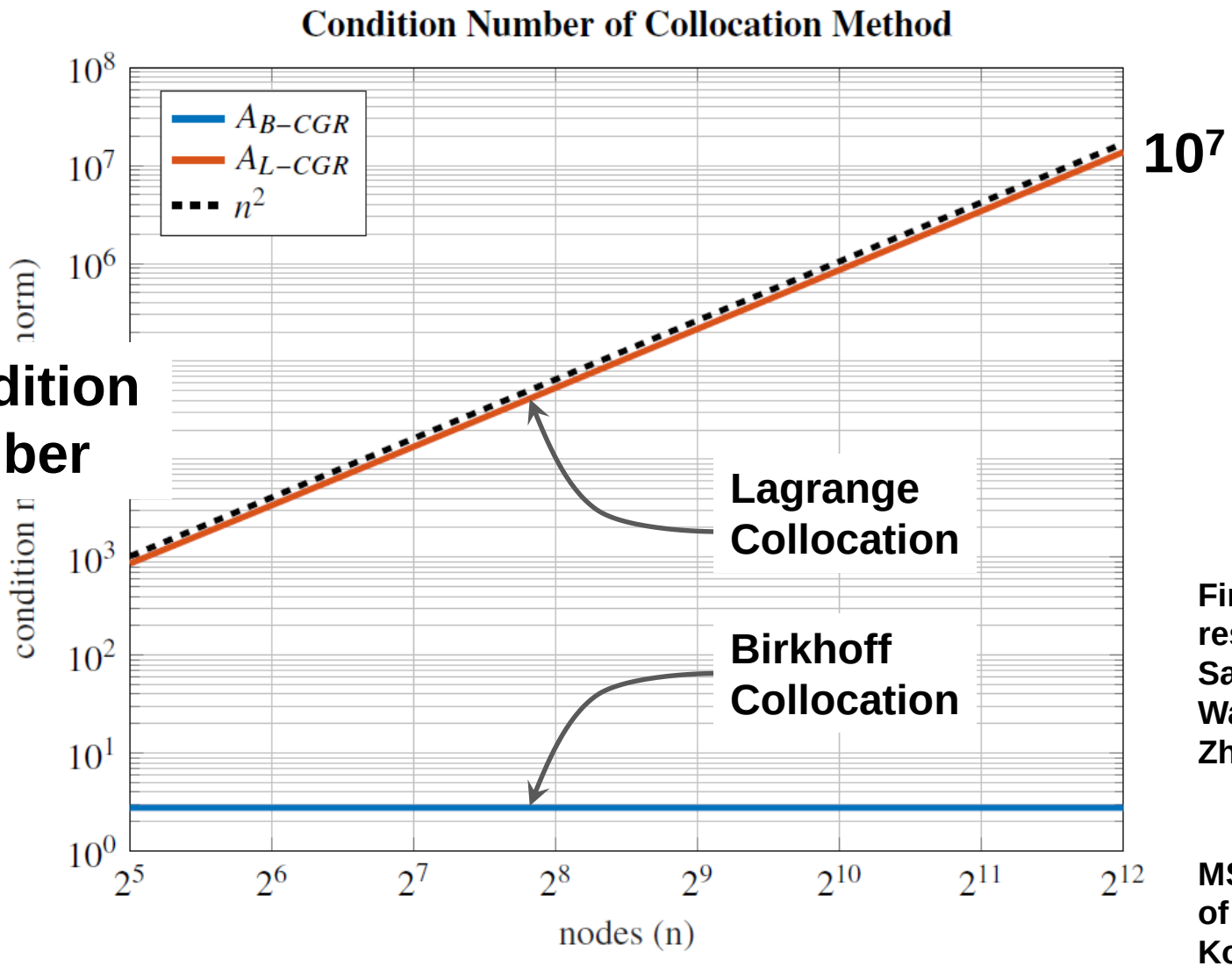
The Birkhoff interpolation polynomial p of u can be uniquely determined by

$$p(x) = u(-1)B_0(x) + \sum_{j=1}^{N-1} u''(x_j)B_j(x) + u(1)B_N(x), \quad x \in [-1,1],$$

if one can find $\{B_j\}_{j=0}^{N} \subseteq \mathbb{P}_N$, such that

$$\begin{aligned}
&B_0(-1) = 1, \quad B_0(1) = 0, \quad B_0''(x_i) = 0, \quad 1 \le i \le N-1;\\
&B_j(-1) = 0, \quad B_j(1) = 0, \quad B_j''(x_i) = \delta_{ij}, \quad 1 \le i, j \le N-1;\\
&B_N(-1) = 0, \quad B_N(1) = 1, \quad B_N''(x_i) = 0, \quad 1 \le i \le N-1.
\end{aligned}$$

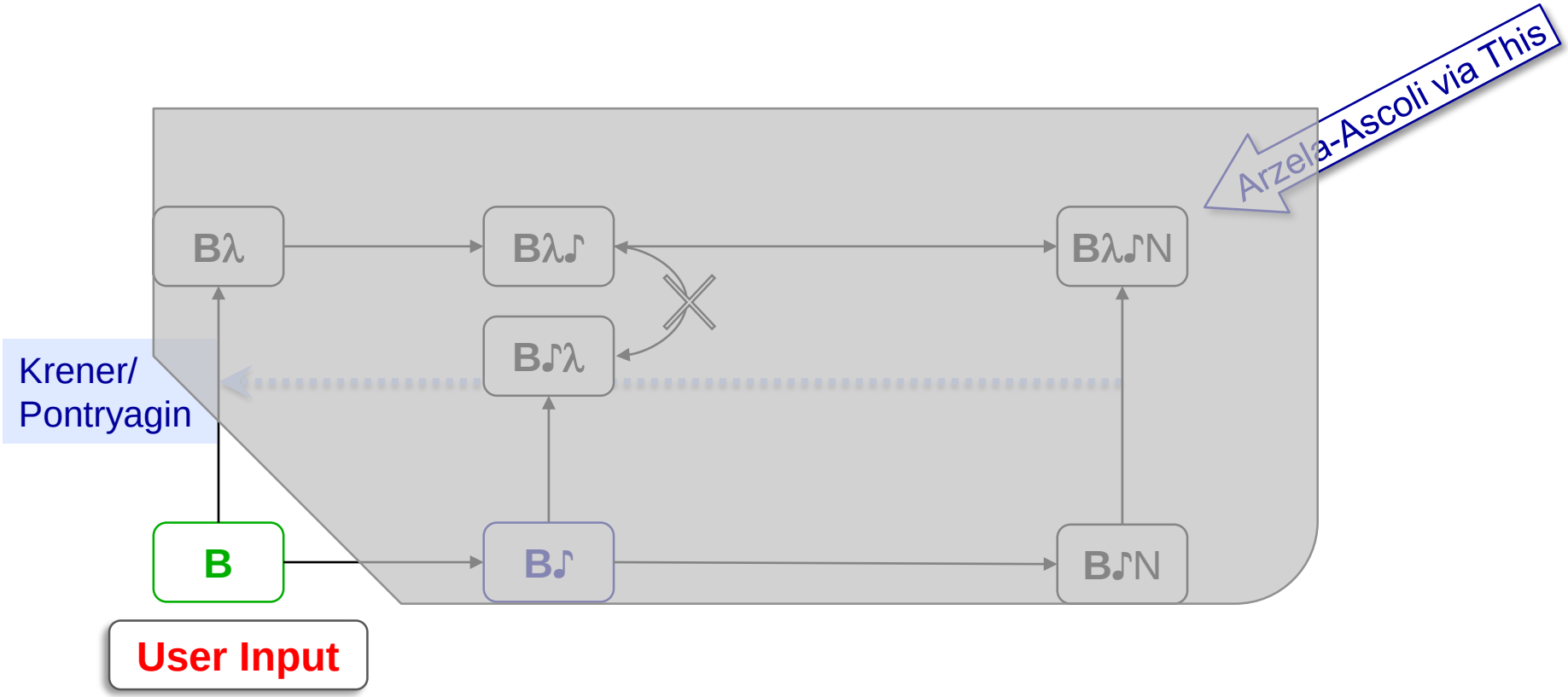
Yet Another Birkhoff "Magic"



First results by Samson, Wang & Zhao, 2014

MS Thesis of Capt Nick Koeppen

A “Turing Test” for Optimal Control Programming



This is DIDO©: A MATLAB® Toolbox for Solving Optimal Control Problems

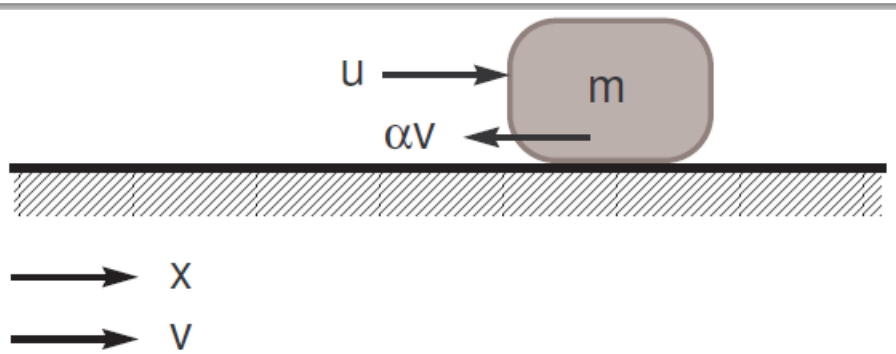
Not so shameless ad: proof of existence that this is do-able.

Illustrative Problem –1

Gong's Challenge Problem:

$$\mathbf{x} = (x, v) \in \mathbb{R}^2, \quad u \in \mathbb{R}$$

$$(G) \begin{cases} \text{Minimize} & J[\mathbf{x}(\cdot), u(\cdot)] = \displaystyle\int_0^1 v(t)u(t)\, dt \\ \text{Subject to} & \dot{x}(t) = v(t) \\ & \dot{v}(t) = -v(t) + u(t) \\ & v(t) \geq 0 \\ & 0 \leq u(t) \leq 2 \\ & (x(0), v(0)) = (0, 1) \\ & (x(1), v(1)) = (1, 1) \end{cases}$$

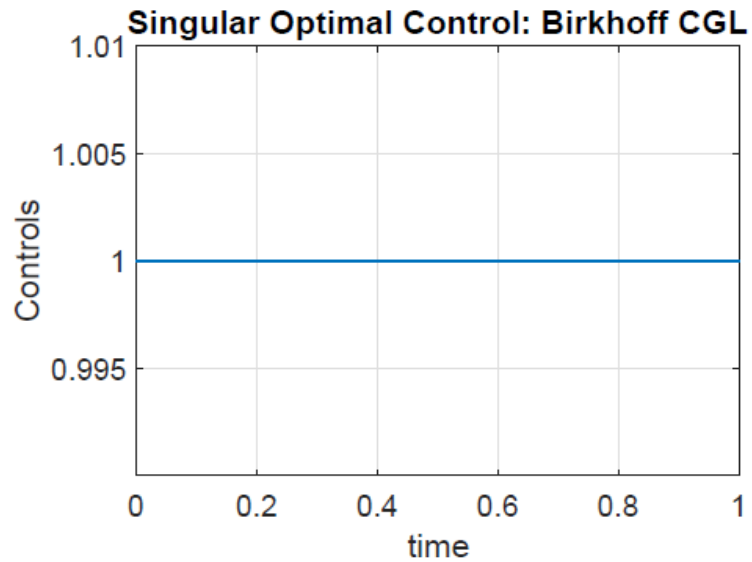


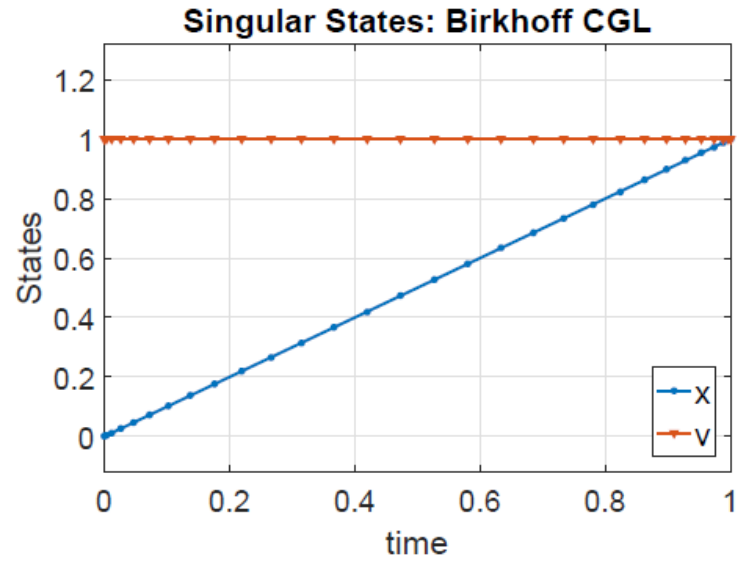
Solution via "Low-Fi" Krener:

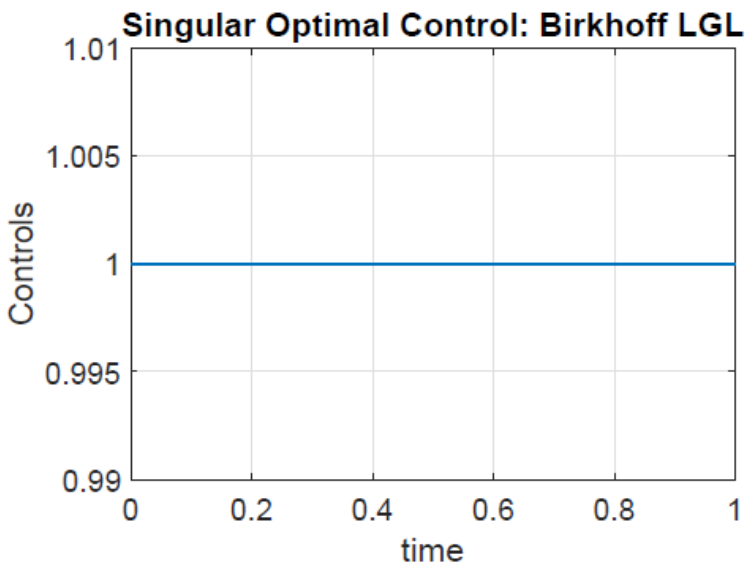
$$\begin{aligned} x^*(t) &= t \\ v^*(t) &= 1 \\ u^*(t) &= 1 \end{aligned}$$

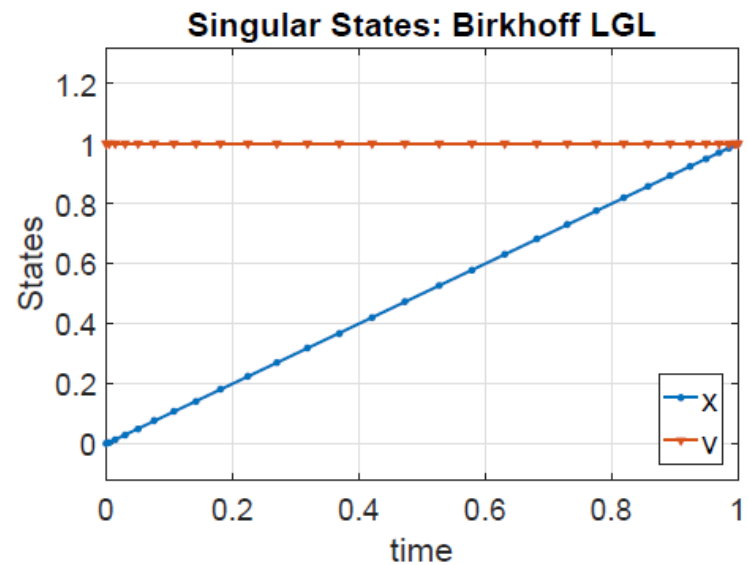
$$\begin{aligned} \lambda_x(t) &= -2 \\ \lambda_v(t) &= -1 \end{aligned}$$

Solution Via Birkhoff



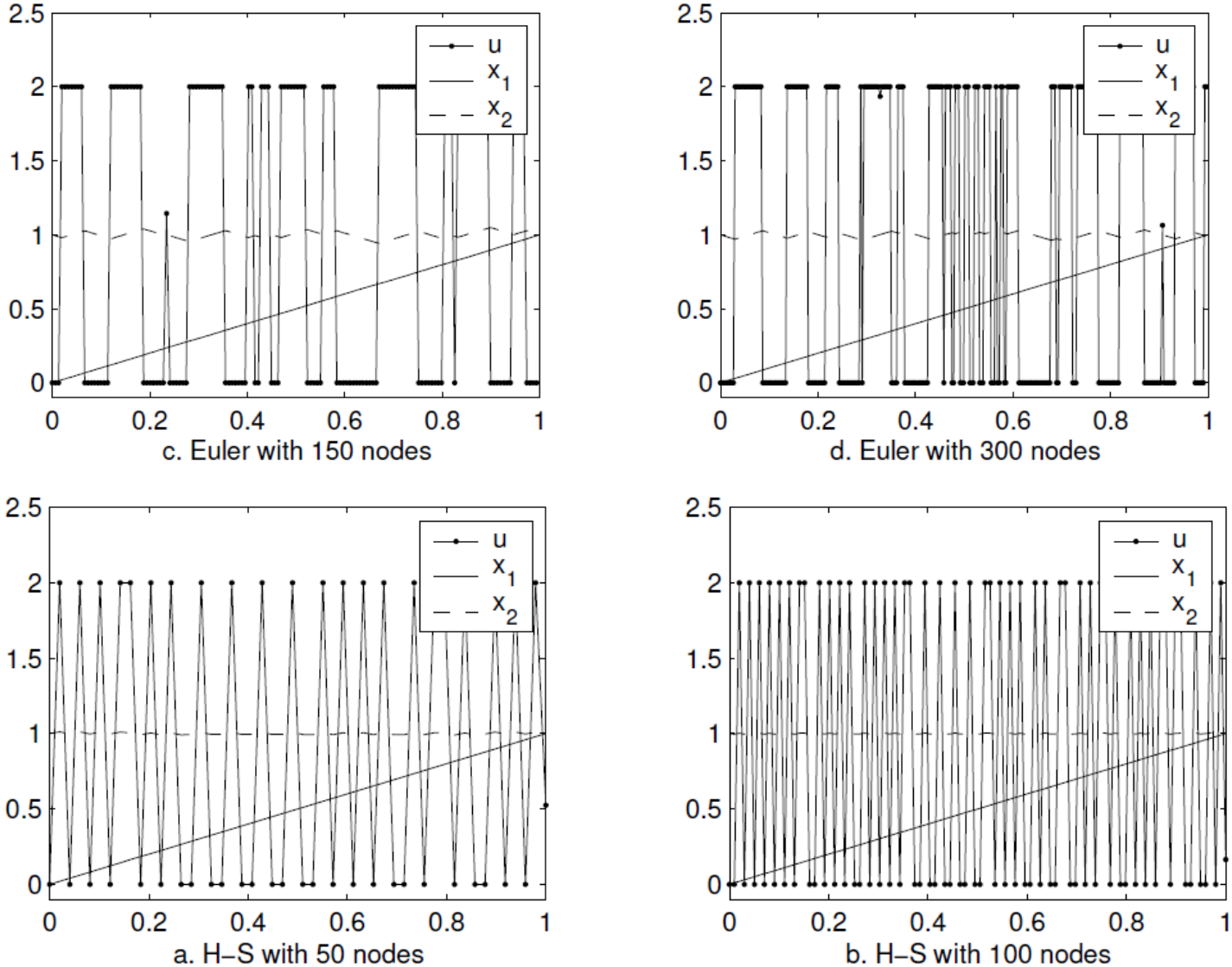






Numerics by Ron Proulx

“Solution” via Runge-Kutta



Converges in Lebesgue, not Sobolev norm

Numerics by Qi Gong

Illustrative Problem –215

- **A DAE Challenge Problem**
  - Betts, Campbell et al (2016, 2018)

$$J(x,u) = \int_0^T cu^2 + d(x - L\sin(t+\alpha))^2 + d(y - L\cos(t+\alpha))^2\, dt \qquad (1)$$

subject to the constraints:

"Index 3 DAE"

$$(D): \begin{cases} \ddot{x} = -\lambda x - a\dot{x} + uy \\ \ddot{y} = -\lambda y - a\dot{y} - g - ux \\ 0 = x^2 + y^2 - L^2 \\ x(0) = 0 \;\; y(0) = L \;\; \dot{x}(0) = 0 \;\; \dot{y}(0) = 0 \end{cases} \qquad (2)$$

- **Not solvable via several software (e.g., SOCX)**
  - INDEPENDENTLY VERIFIED

"Low-Fi" Krener Conditions for the "DAE" Problem

- **The DAE challenge problem is also singular**
  - Krener-type necessary conditions can be easily derived:
  - Repeated diff … GLC, "Goh-Robbins"

$$\frac{d^k}{dt^k}(\partial_u H) = 0$$

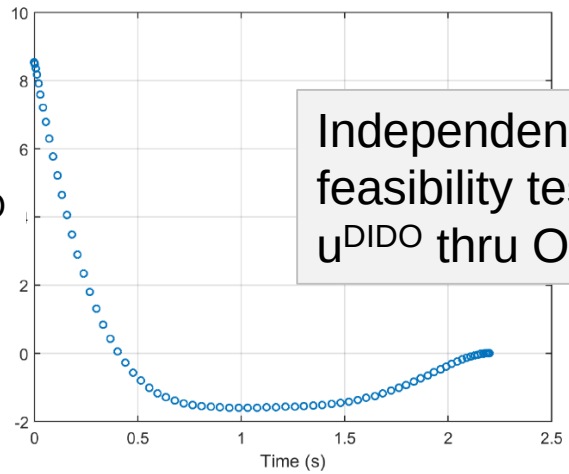
One of many reasons why we want convergence in Sobolev spaces

Sample singular integral of motion

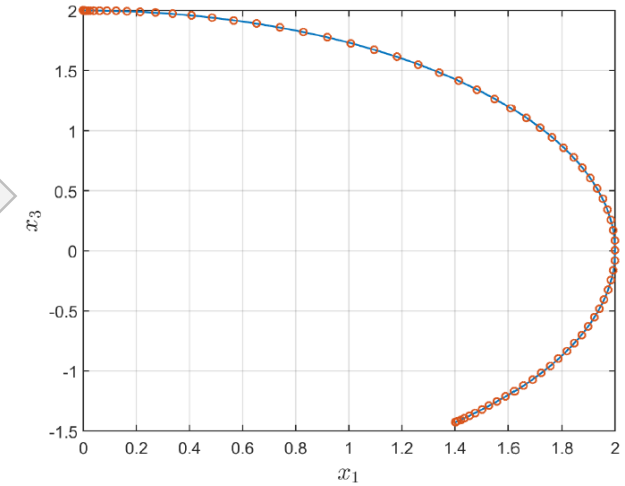
$$-\lambda_{x_2} x_1 - \lambda_{x_4} x_3 = 0$$

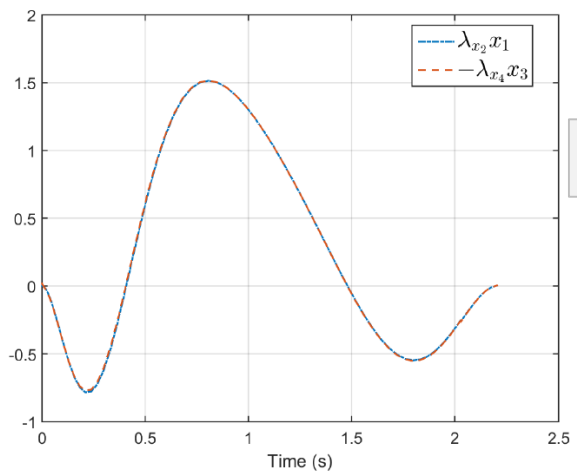
Solution via DIDO w/ Independent V&V

u^{DIDO}



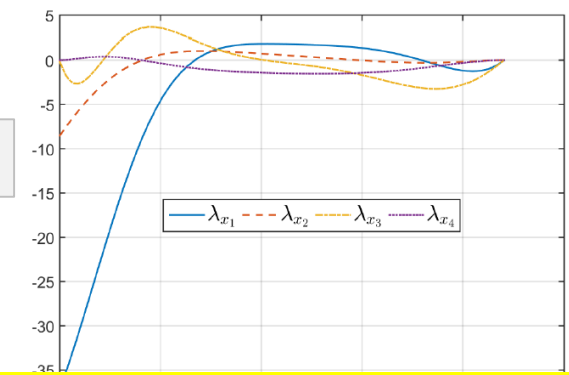
Independent primal feasibility test: propagate u^{DIDO} thru ODE





Dual feasibility tests

$$-\lambda_{x_2} x_1 - \lambda_{x_4} x_3 = 0$$



Proof of the possible!

- **DIDO requires no *a priori* "index reduction"**
  - DIDO solution agrees w/ SOCX solution generated after a laborious process of index reduction

Numerics by Mark Karpenko

Do The Principles Hold For "Regular" Problems?

- **Short answer is yes!**
  - Krener's HMP was an "easy way" to prove flaws in some arguments
- **Pontryagin's Principle (non-singular problems)**
  - Hamiltonian minimization condition
    - = NLP within the collection of necessary conditions
    - Pointwise condition
  - Discretization generates necessary conditions for a necessary condition!
    - Via KKT
- **Vast number of "nuances" that separate optimal control from "large scale NLPs"**
  - HMC
  - States and controls must be treated differently; NLP is agnostic to this distinction
  - x_{k+1} and x_k are "near each other" via Lipschitz; in NLP they are "random"
  - Scaling each x_k generates "additional" dynamics that are not in the optimal control problem
    - Scaling and balancing in optimal control problems
  - Scaling u_k can generate accuracy problems in optimality
  - Many more!

Take Aways

- **A significant body of work beyond NLP theory/tools remains to be done**
  - A vast number of theoretical issues are open
    - PhD topics (& beyond) for research!
- **"Turing software" based on some/many necessary conditions are possible**
  - DIDO = proof of the possible
- **Presented theory and tools are actively used by NASA, DoD and some industrial sectors**
  - Certificate program at the Naval Postgraduate School (Cert # 299)
- **"Hard" problems are not as hard (as advertised)**
  - Practical problems are mathematically rich: "singular," "high-index DAEs," "highly constrained" etc.
- **Computational optimal control based on necessary conditions is unquestionably better than one without it**
  - Necessary conditions might actually be necessary!

Thank You!