Consistent Approximations in Optimization

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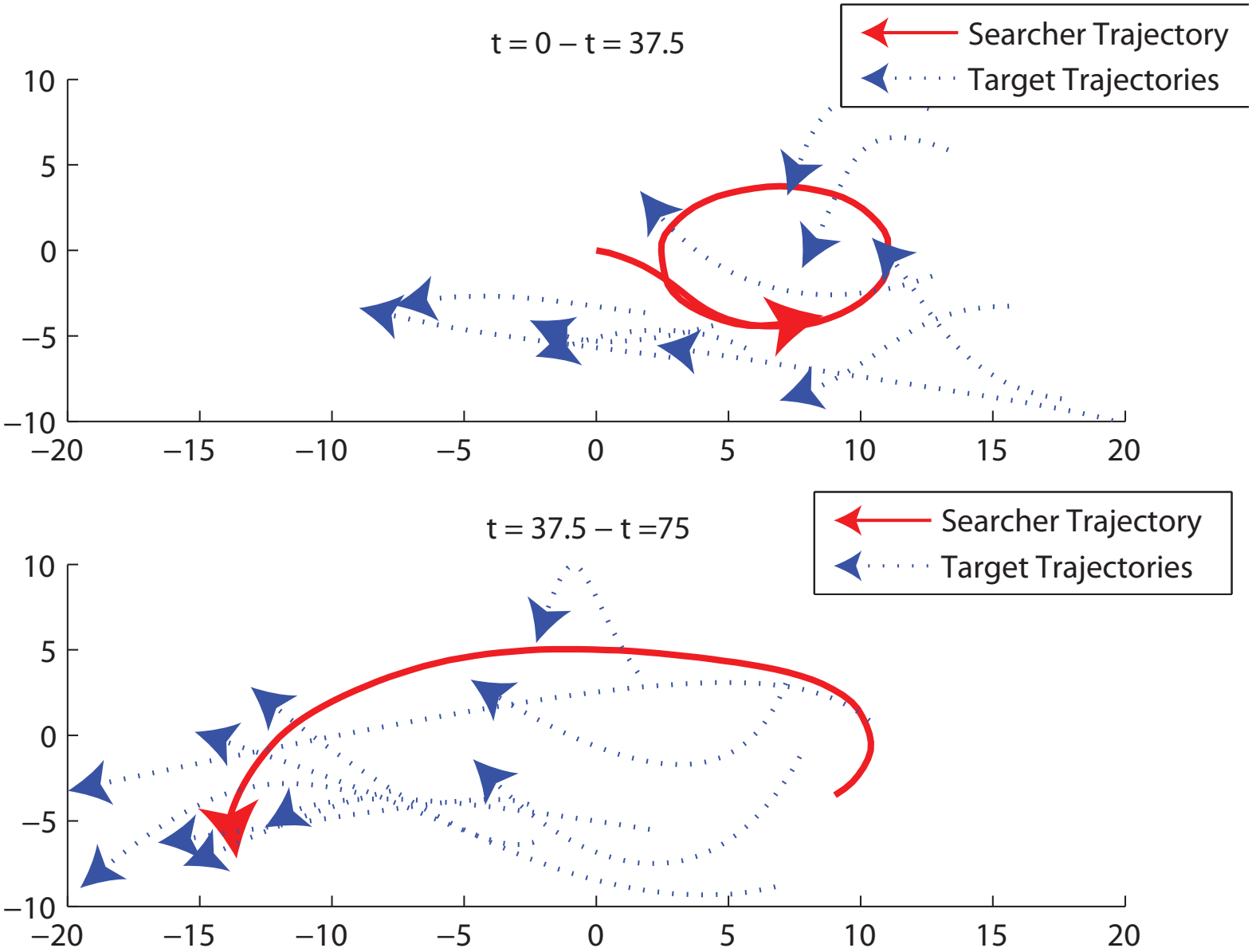
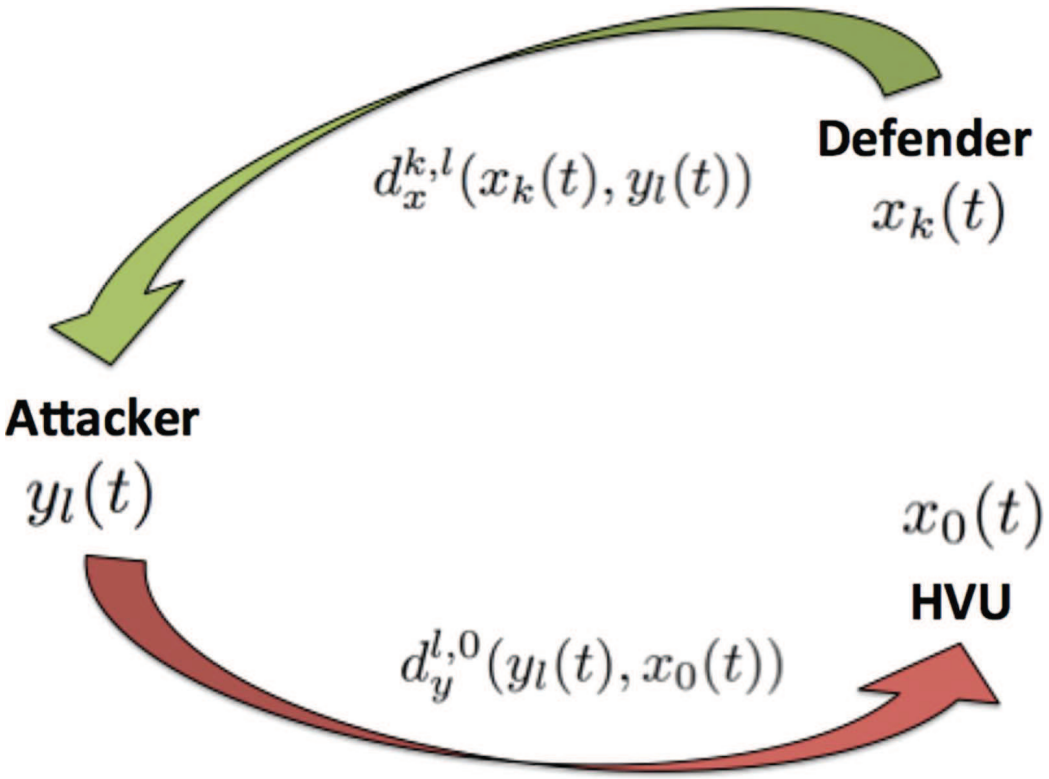


Phelps, Royset & Gong, "Optimal Control of Uncertain Systems using Sample Average Approximations," SIAM J. Control and Optimization, 2016

Stone, Royset & Washburn, Optimal Search for Moving Targets, Springer, 2016

Maximize probability of HVU survival



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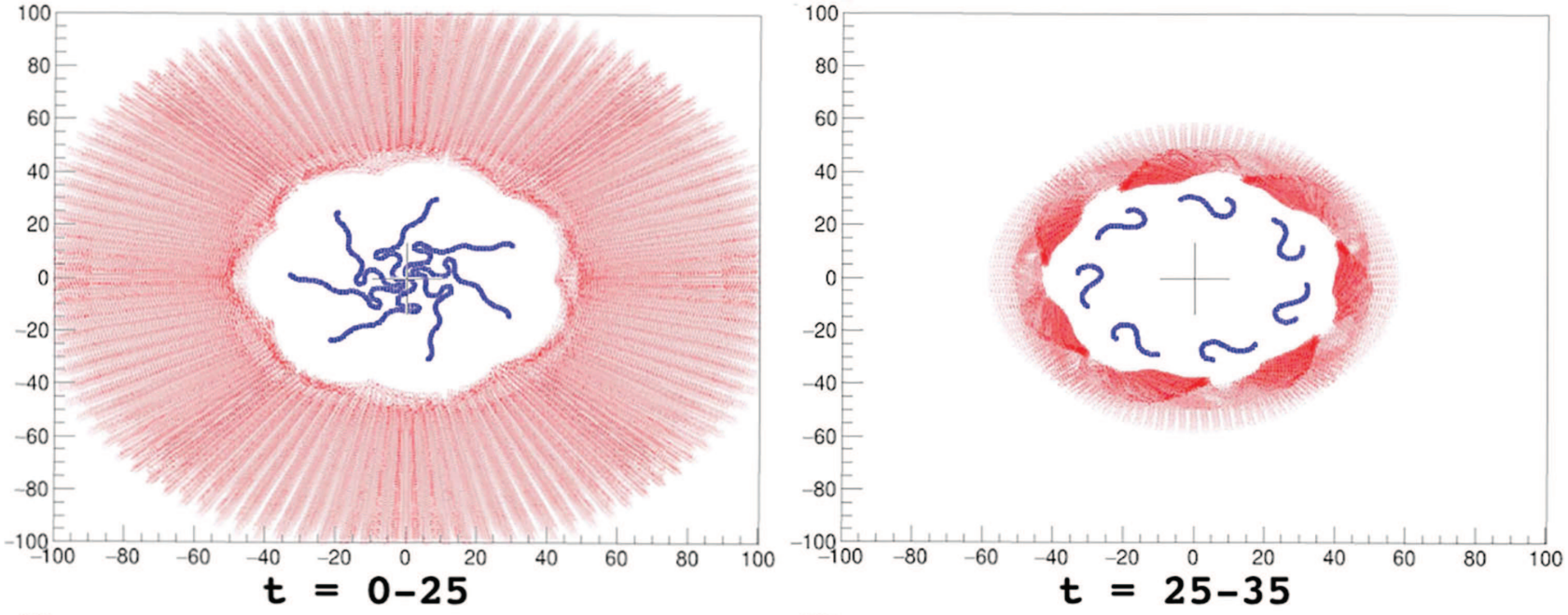

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Setting for presentation

(X, d) metric space
$f^\nu, f : X \to [-\infty, \infty]$, usually lower semicontinuous (lsc)

$$\begin{aligned} \text{Actual problem:} \quad & \underset{x \in X}{\text{minimize}}\, f(x) \\ \text{Approximating problem:} \quad & \underset{x \in X}{\text{minimize}}\, f^\nu(x) \end{aligned}$$

Constraints often handled abstractly:
Setting objective function to ∞ if x infeasible (wlog)

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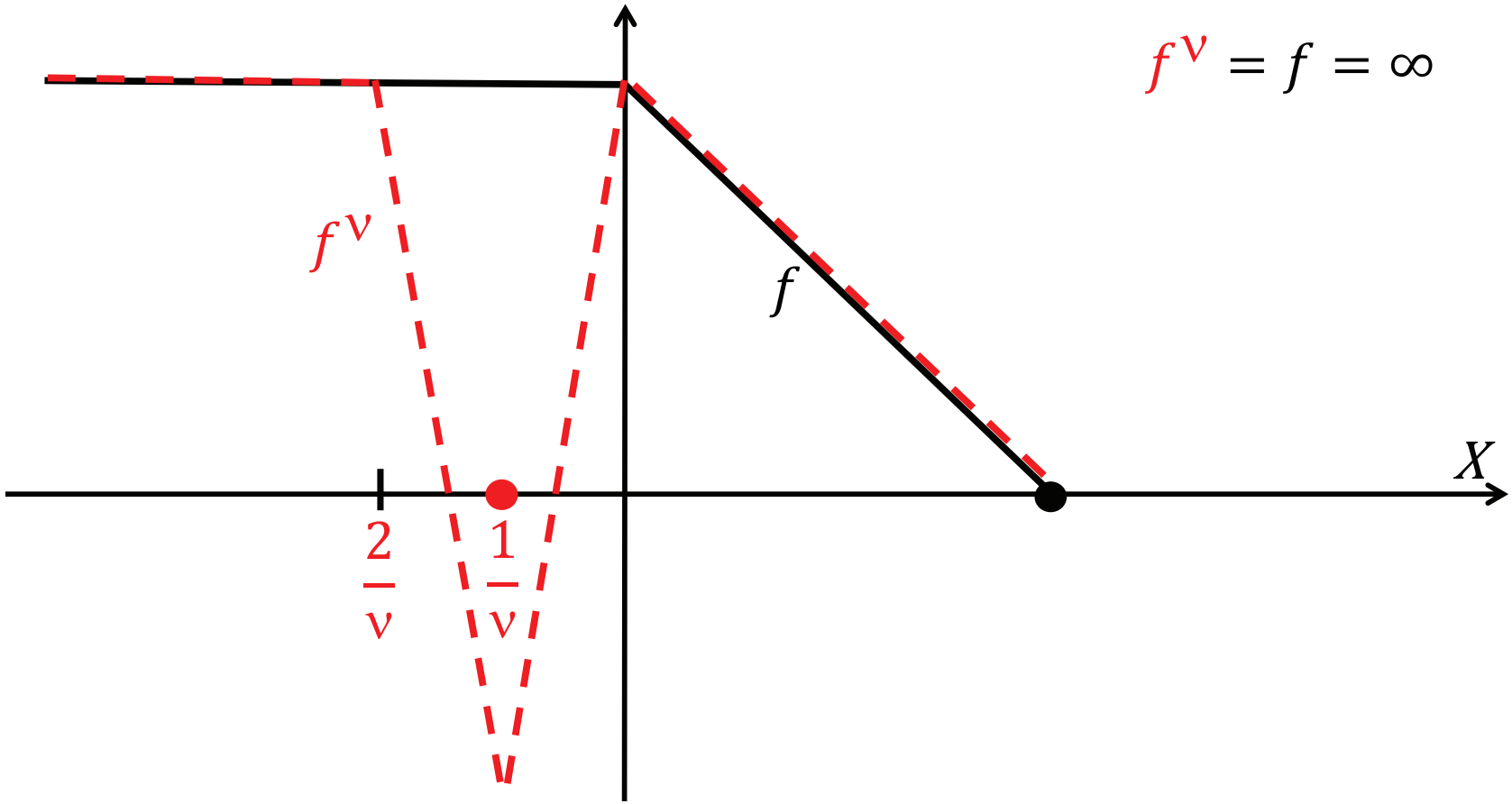
What constitutes a consistent approximation?

Level 0: convergence of minimizers, minima

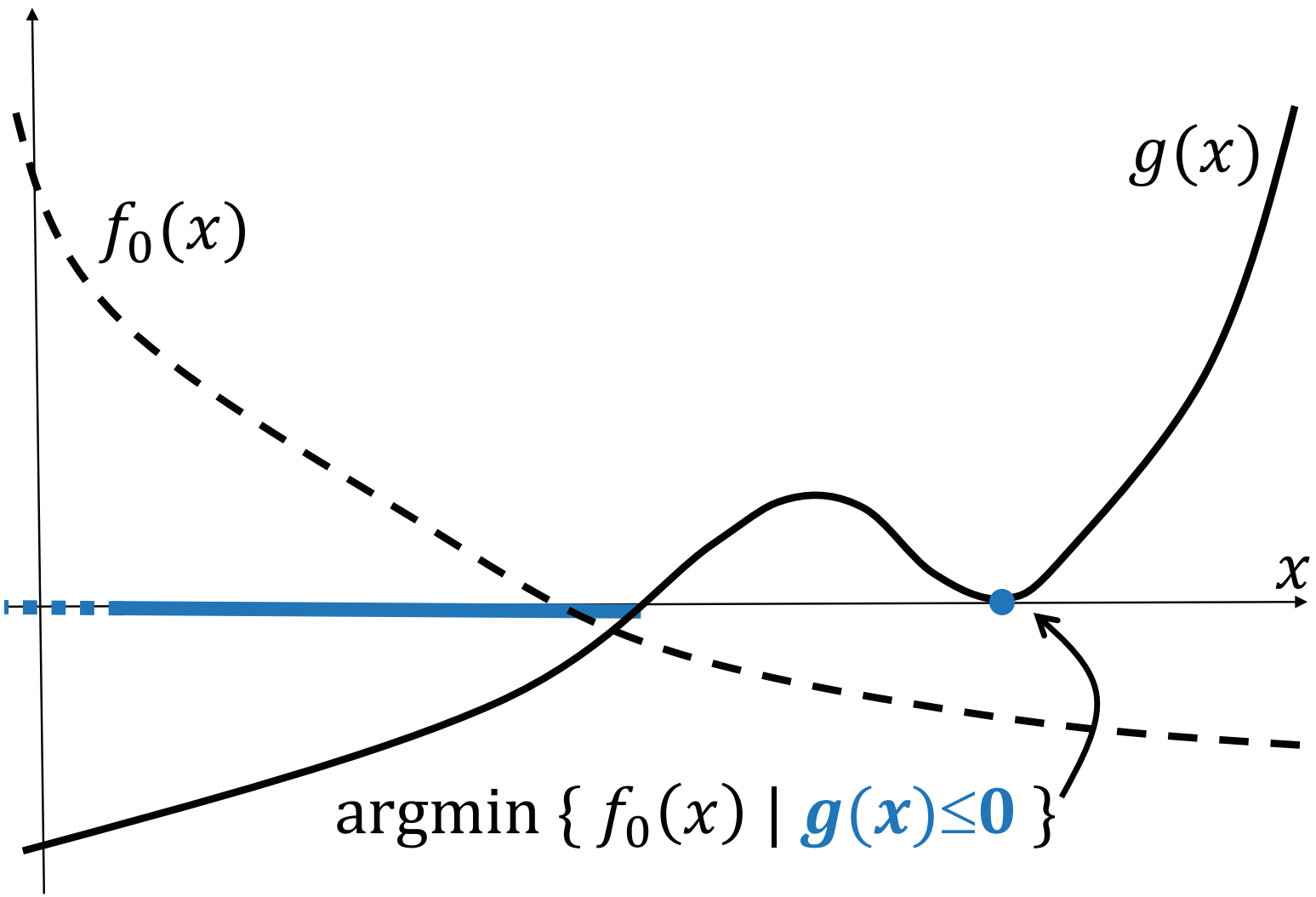
Level 1: convergence of first-order stationary points

Would pointwise convergence suffice?



Pointwise convergence not sufficient for convergence of minimizers

What about uniform convergence?



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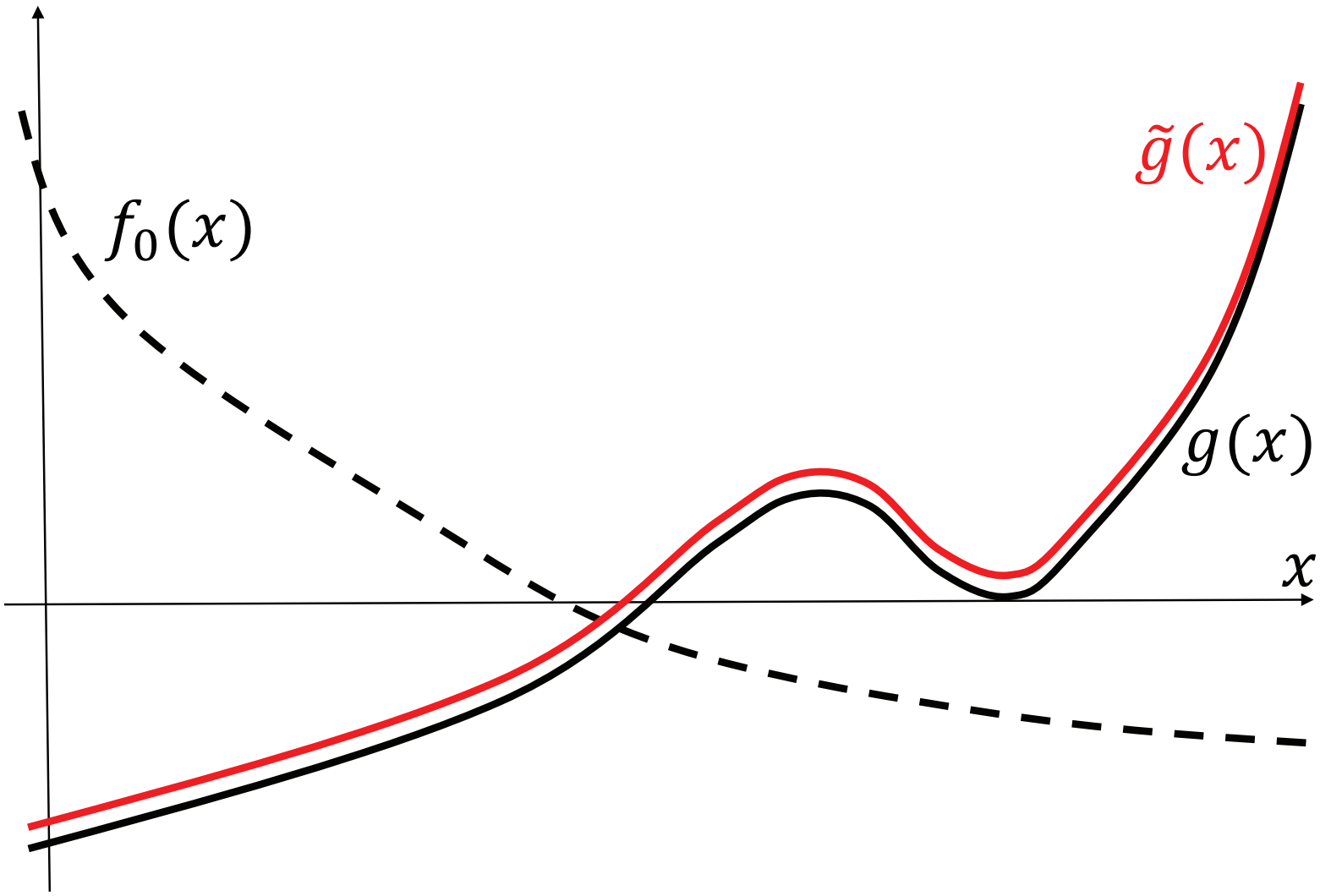
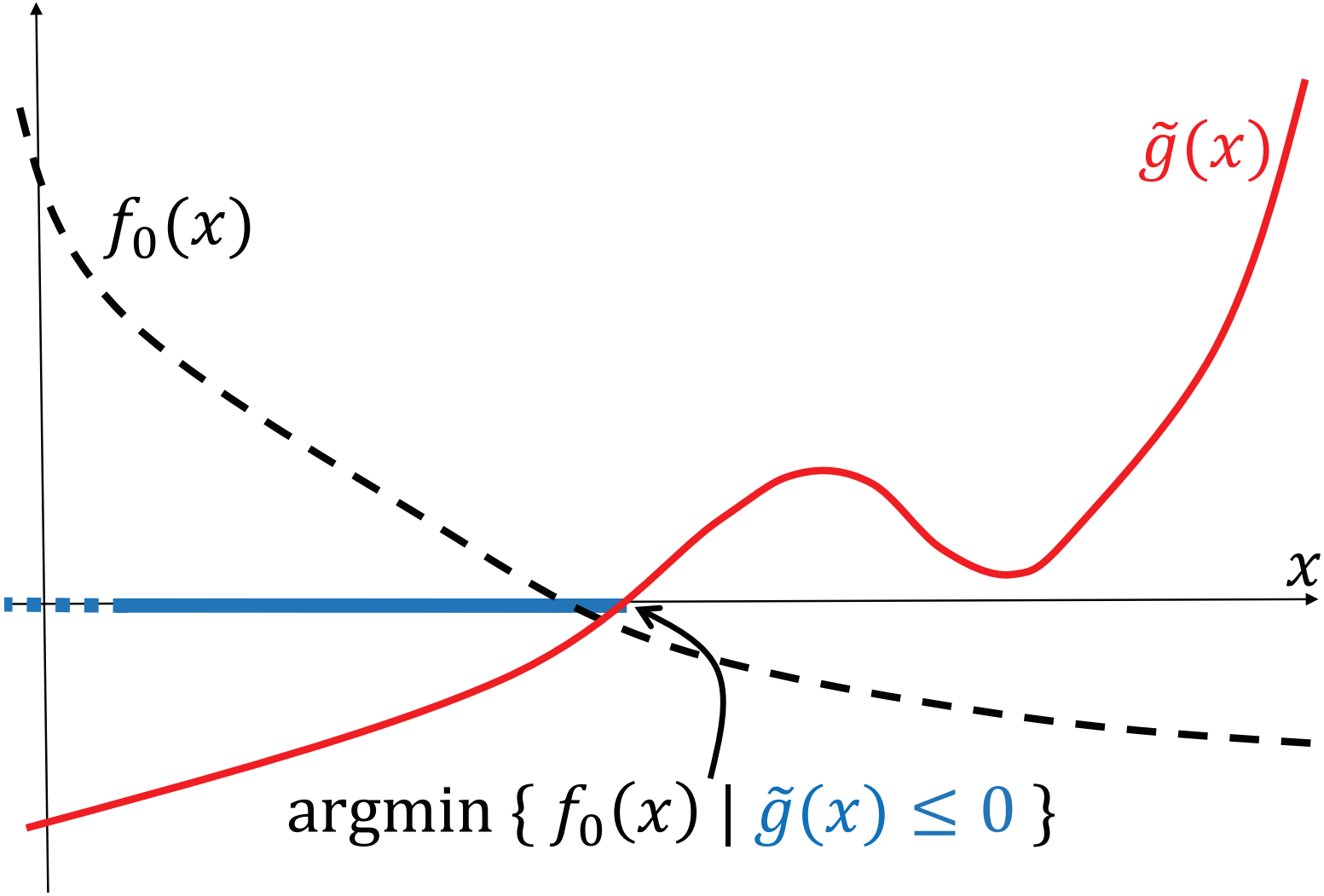
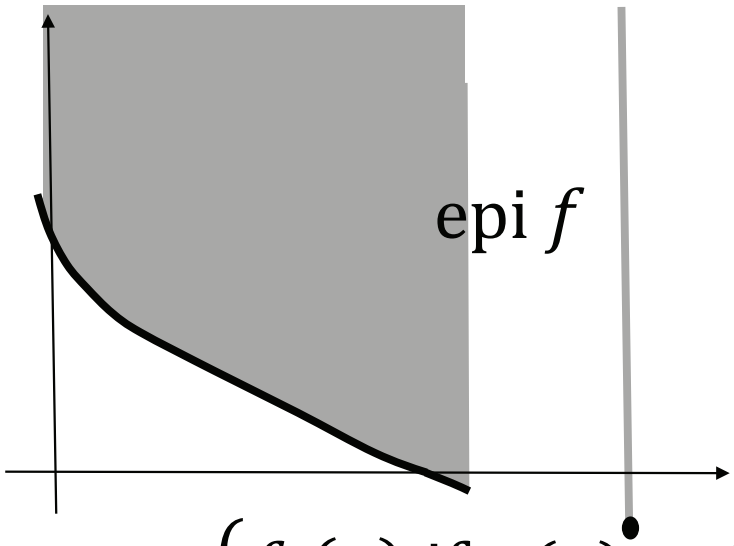
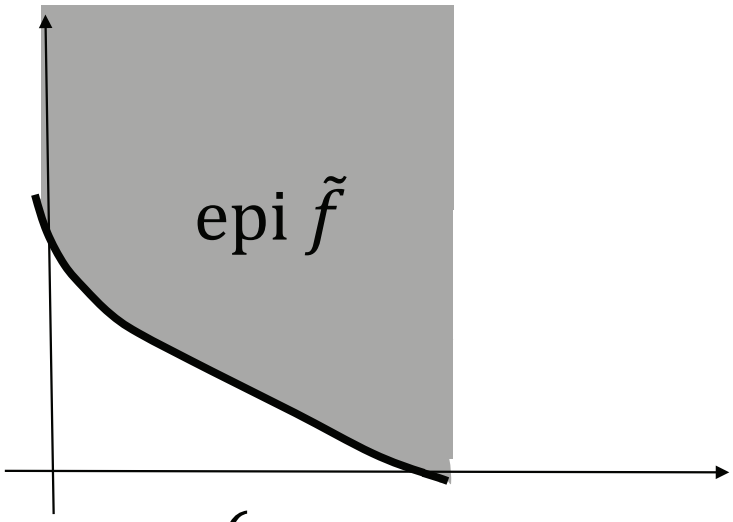


Uniform "approximation," but large error in argmin



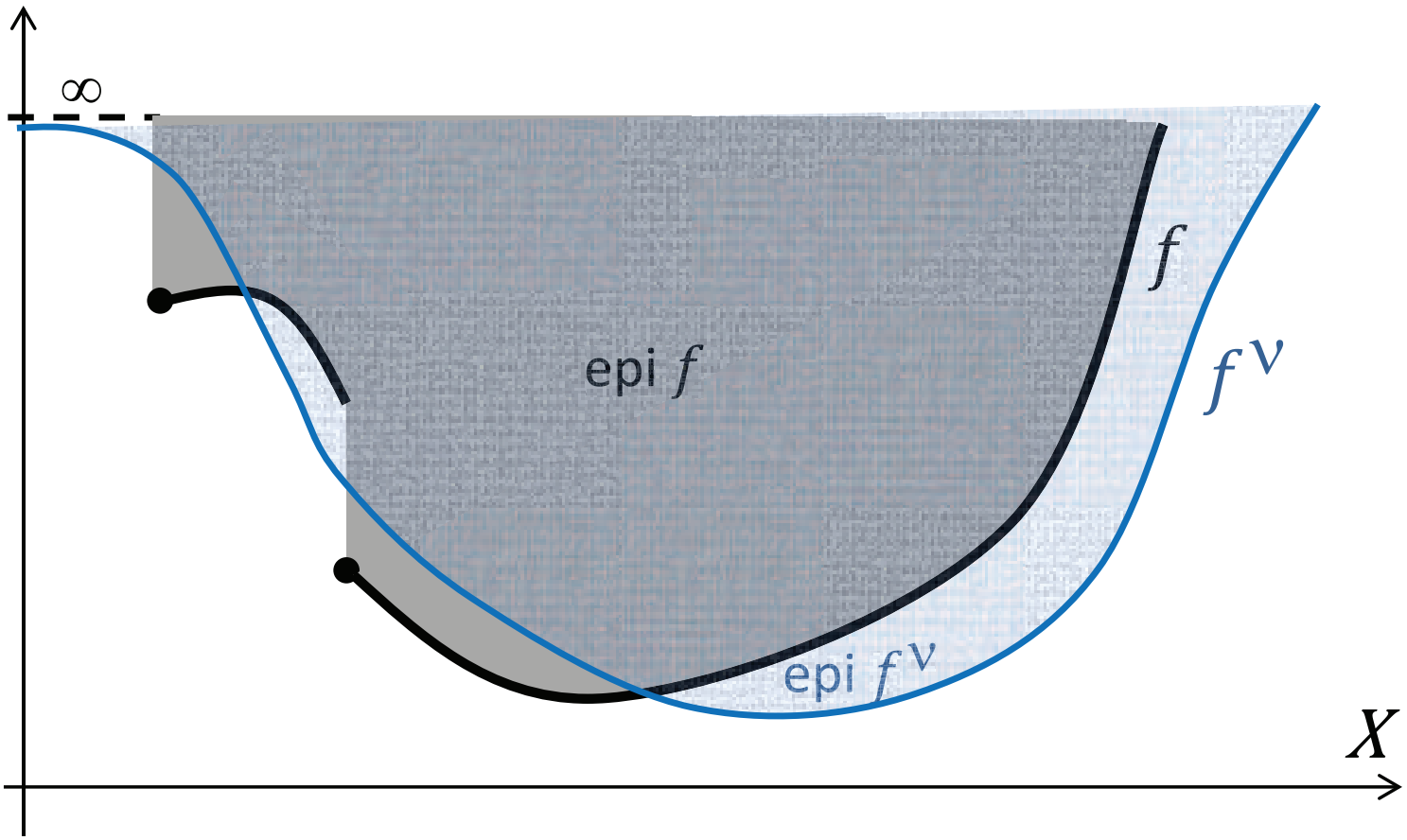
Passing to epigraphs of the effective functions





$$f(x) = \begin{cases} f_0(x) & \text{if } g(x) \leq 0 \\ \infty & \text{otherwise} \end{cases} \qquad \tilde{f}(x) = \begin{cases} f_0(x) & \text{if } \tilde{g}(x) \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

Epi-convergence

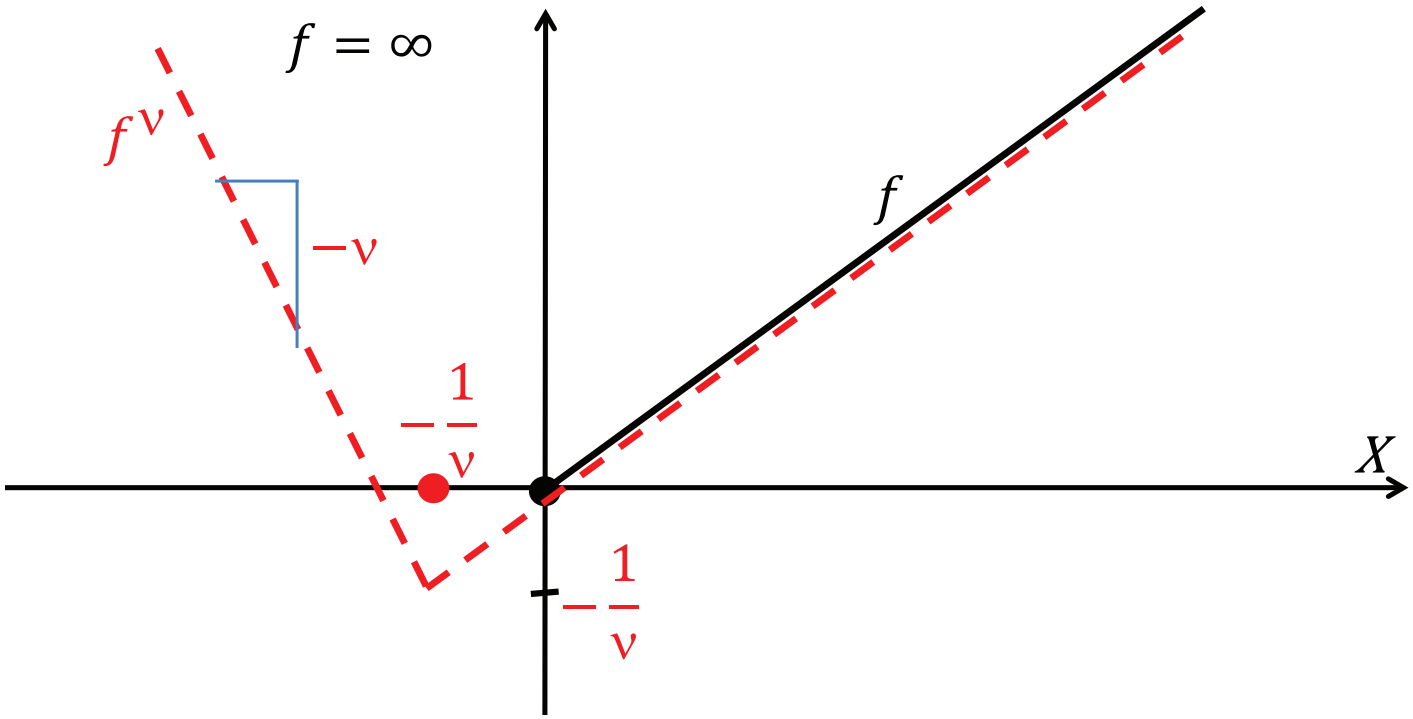


f^ν epi-converges to f $\iff$ epi f^ν set-converges to epi f

Main consequence:

f^ν epi-converges to f and $x^\nu \in \operatorname{argmin} f^\nu \to \bar{x} \implies \bar{x} \in \operatorname{argmin} f$

Approximation of constraints



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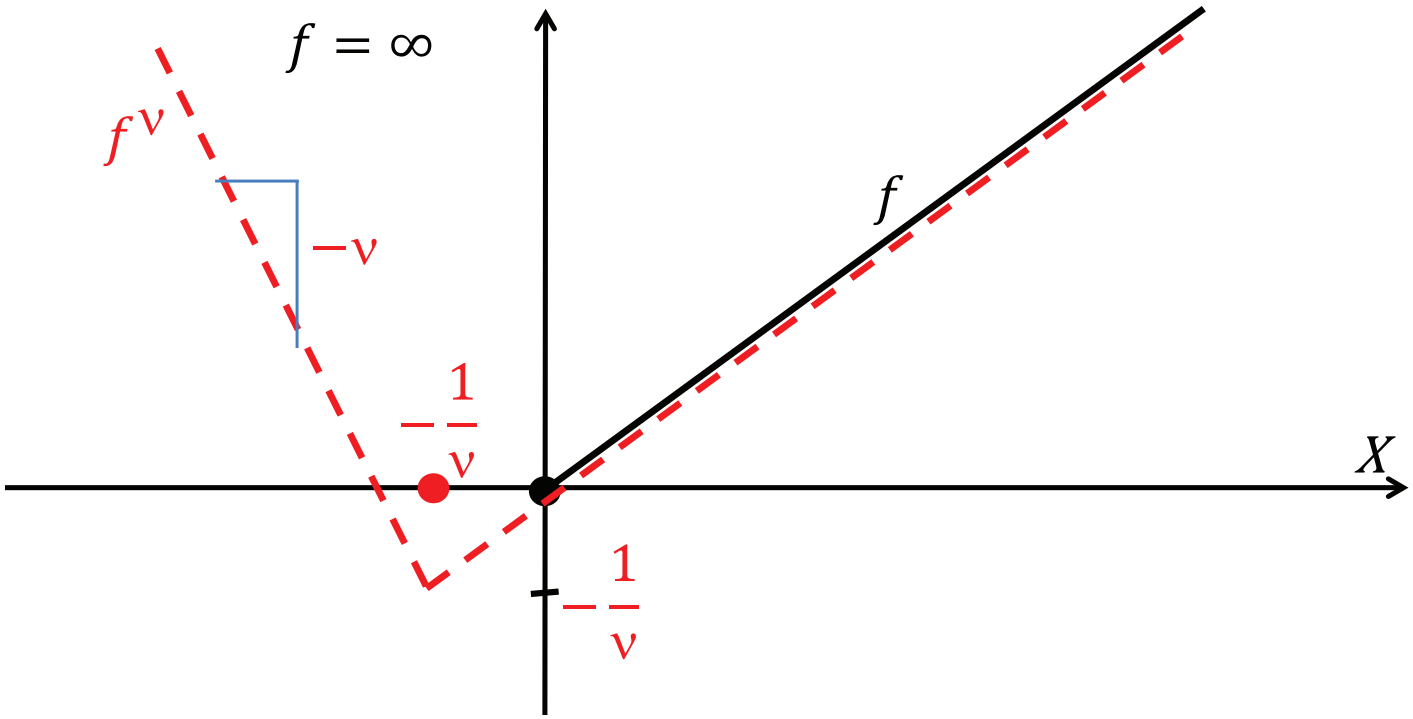

If C^ν set-converges to C and f_0 continuous, then

$$f^\nu(x) = \begin{cases} f_0(x) & \text{if } x \in C^\nu \\ \infty & \text{otherwise} \end{cases} \quad \text{epi-conv to } f(x) = \begin{cases} f_0(x) & \text{if } x \in C \\ \infty & \text{otherwise} \end{cases}$$

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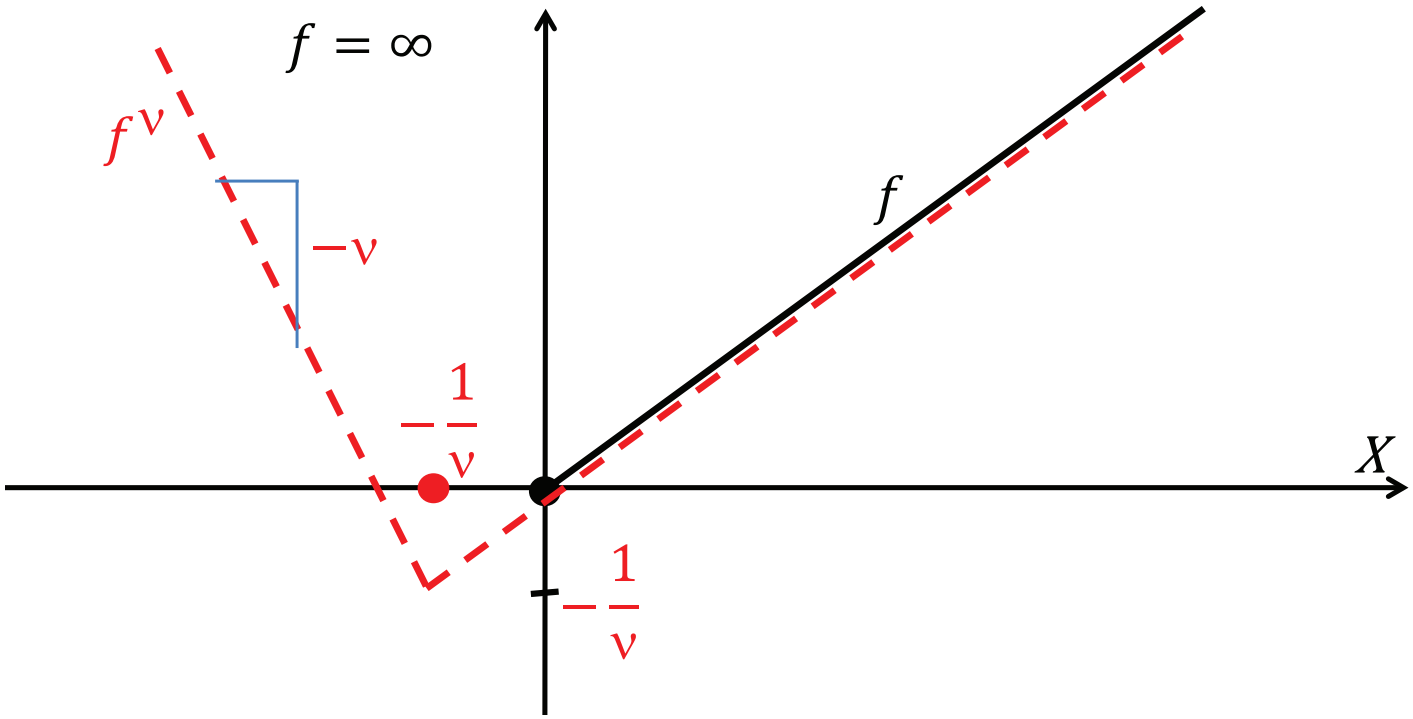

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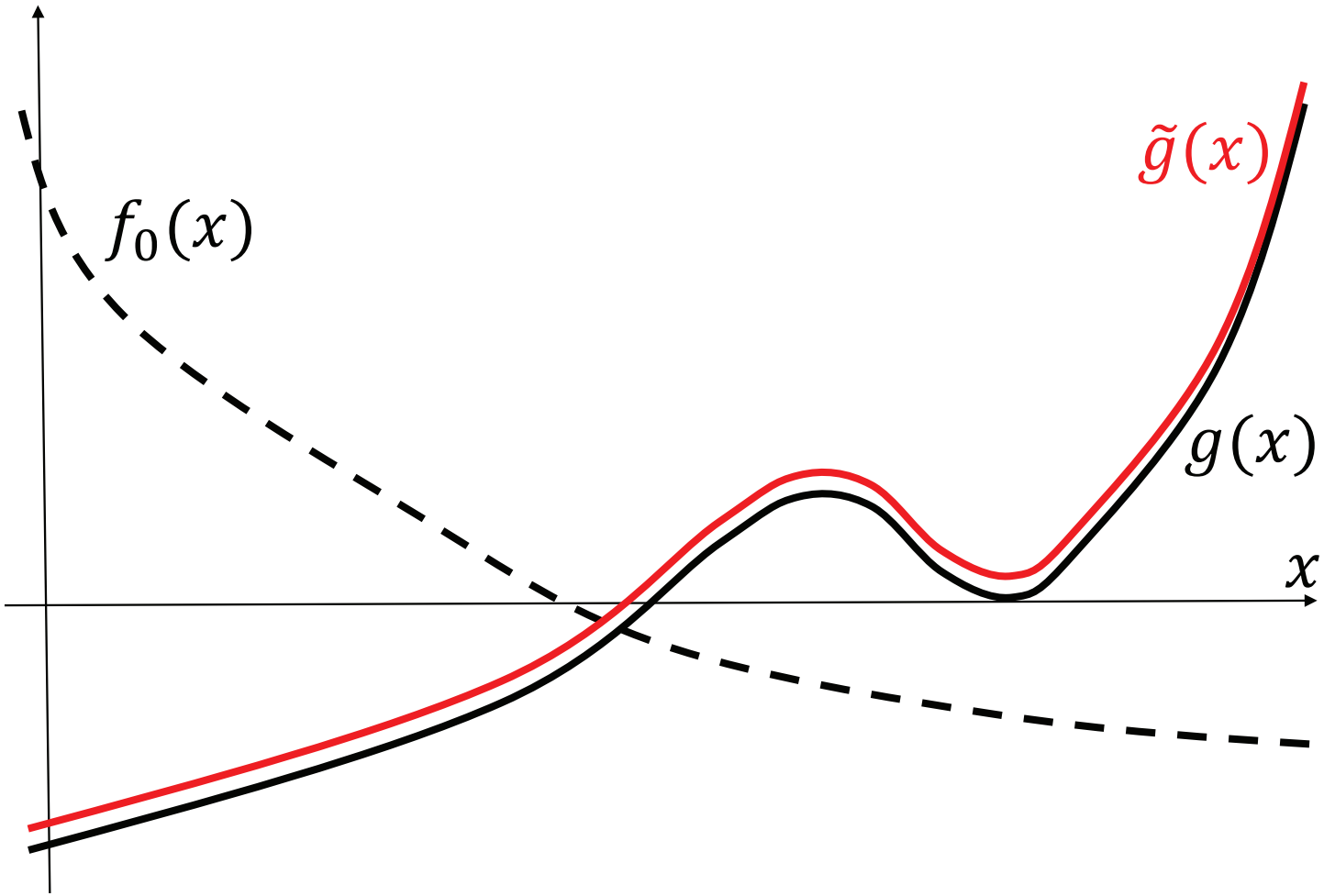
Example: $C^1, C^2, \ldots$ dense in $C = X \implies C^\nu$ set-converges to C

Recall failure under uniform convergence

What can be done in this case?



Constraint softening

$$\underset{x \in X}{\text{minimize}}\, f_0(x) \text{ subject to } g_i(x) \leq 0, \;\; i = 1, \ldots, q$$

$$\sup_{x \in X} |f_0^\nu(x) - f_0(x)| \leq \alpha^\nu \quad \text{and} \quad \sup_{x \in X} \max_{i=1,\ldots,q} |g_i^\nu(x) - g_i(x)| \leq \alpha^\nu$$

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$$\underset{x \in X, y \in \mathbb{R}^q}{\text{minimize}} \;\; f_0^\nu(x) + \theta^\nu \sum_{i=1}^{q} y_i \;\; \text{subject to} \;\; g_i^\nu(x) \leq y_i, \;\; 0 \leq y_i, \;\; i = 1, \ldots, q$$

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$$\underset{x \in X}{\text{minimize}}\, f_0(x) \text{ subject to } g_i(x) \leq 0,\ \ i = 1, \ldots, q$$

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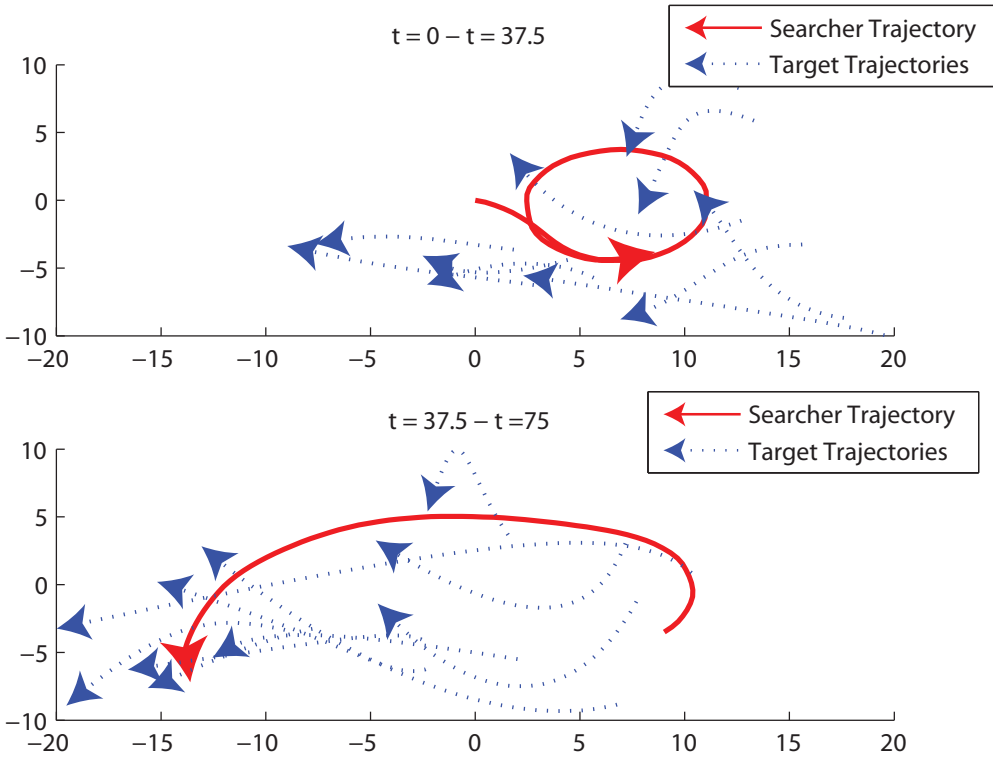
f_0 continuous
g_i lsc, $i = 1, \ldots, q$
$\theta^\nu \to \infty$, $\alpha^\nu \to 0$, $\theta^\nu \alpha^\nu \to 0$

Then, approximation epi-converges to actual

Epi-convergence under sampling and forward Euler



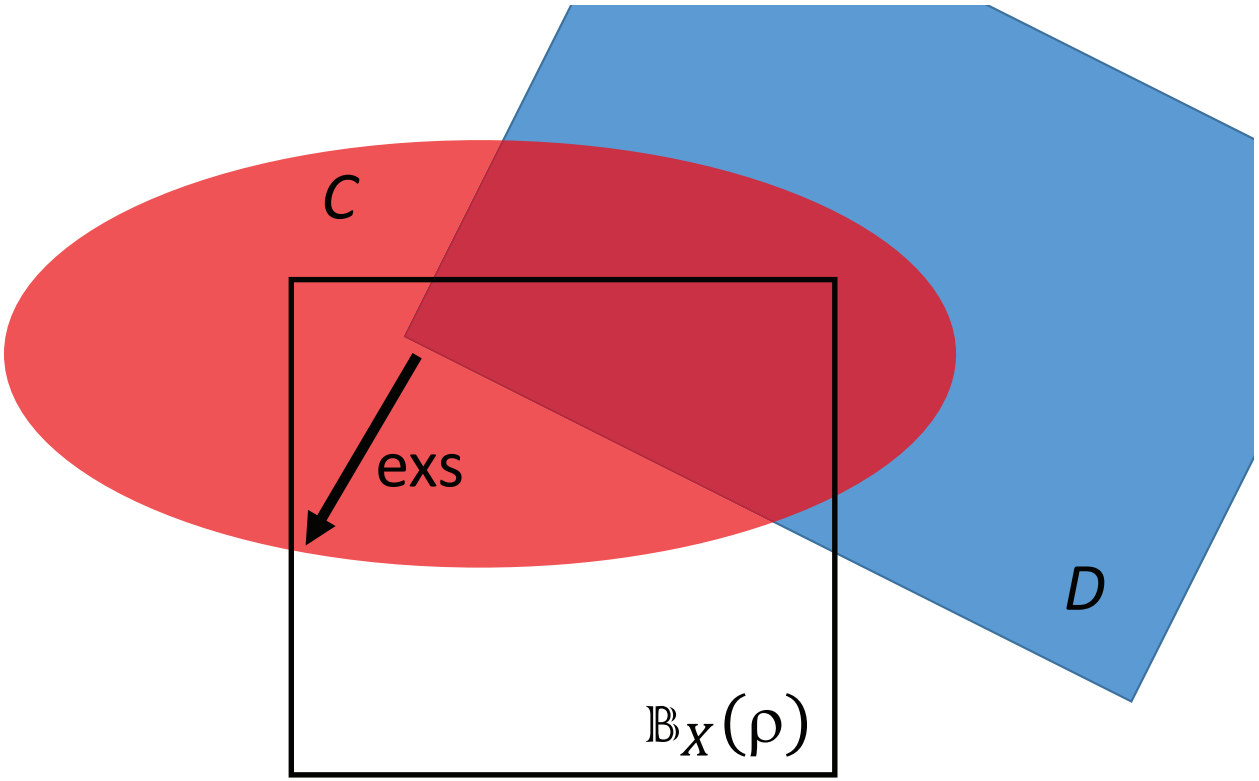
$$\begin{aligned} \text{minimize } J(\xi, u) =& \mathbb{E}\big[F(x(1,\omega),\omega)\big] \\ \text{subject to } \dot{x}(t,\omega) =& f(x(t), u(t), \omega), \quad x(0,\omega) = \xi + x_0(\omega), \quad \forall \omega \end{aligned}$$

Sampling and Forward Euler result in epi-convergence

Phelps, Royset & Gong, "Optimal Control of Uncertain Systems using Sample Average Approximations," SIAM J. Control and Optimization, 2016

Truncated Hausdorff distance between sets

For $C, D \subset X$ (metric space)



$$\hat{dl}_\rho(C, D) = \max\left\{ \operatorname{exs}\left(C \cap \mathbb{B}_X(\rho); D\right),\ \operatorname{exs}\left(D \cap \mathbb{B}_X(\rho); C\right) \right\}$$

Consequence for minima and near-minimizers

For $f, g : X \to [-\infty, \infty]$,

$$|\inf f - \inf g| \leq d\hat{l}_\rho(\operatorname{epi} f, \operatorname{epi} g)$$

$$\operatorname{exs}\left(\varepsilon\text{- argmin } g \cap \mathbb{B}_X(\rho);\ \delta\text{- argmin } f\right) \leq d\hat{l}_\rho(\operatorname{epi} f, \operatorname{epi} g)$$
$$\text{if } \delta > \varepsilon + 2d\hat{l}_\rho(\operatorname{epi} f, \operatorname{epi} g)$$

(product metric is used on $X \times \mathbb{R}$ and ρ large enough)

Replace $>$ by $\geq$ when f and g lsc and X has compact balls

Bounds are sharp

$$\operatorname{exs}\left(\varepsilon\text{- argmin } g \cap \mathbb{B}_X(\rho);\ \delta\text{- argmin } f\right) \leq d\hat{l}_\rho(\operatorname{epi} f, \operatorname{epi} g)$$
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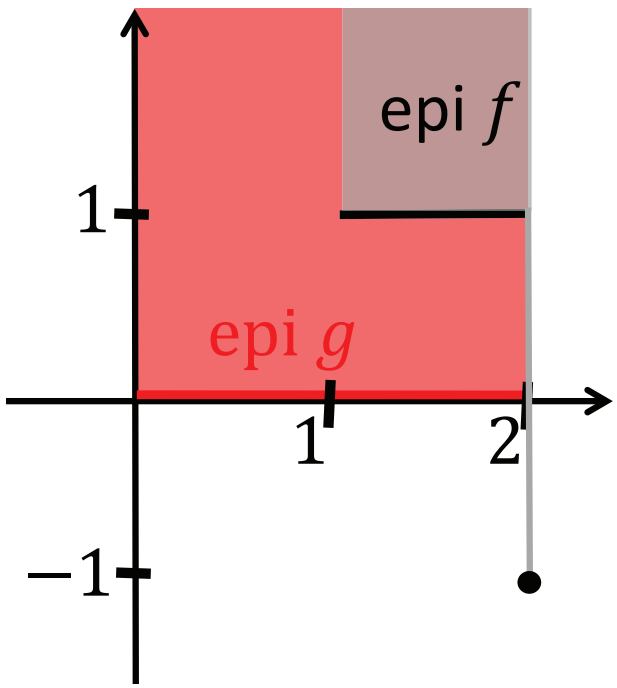
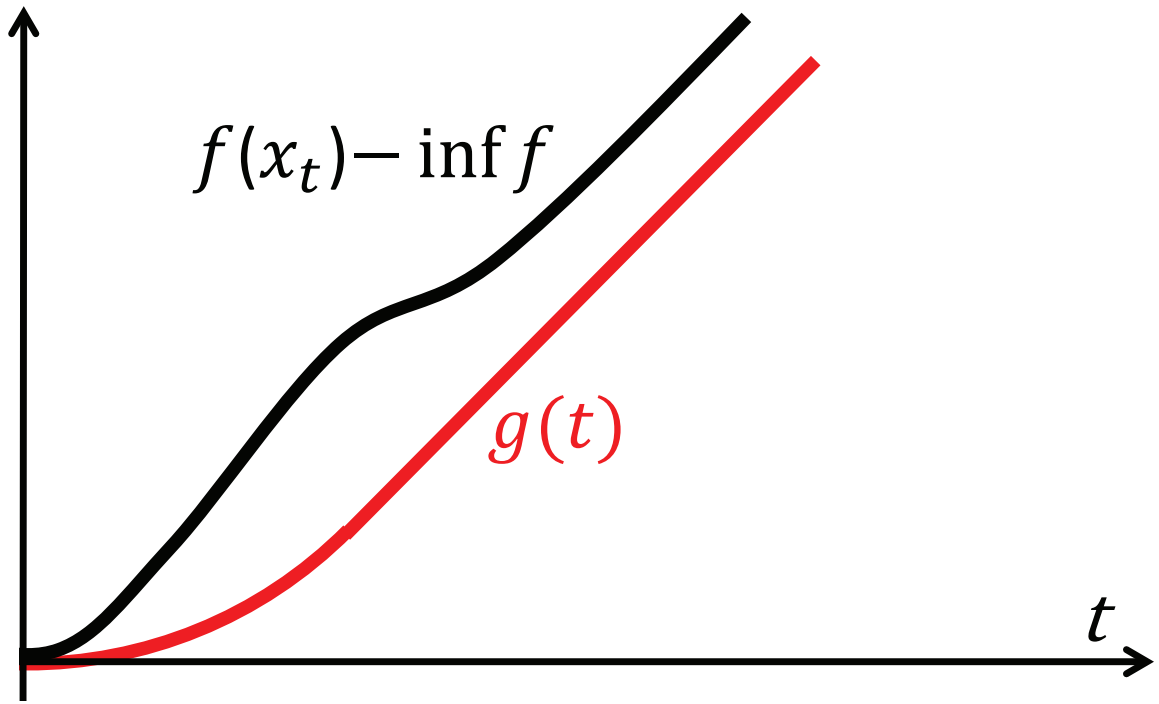


What about minimizers?

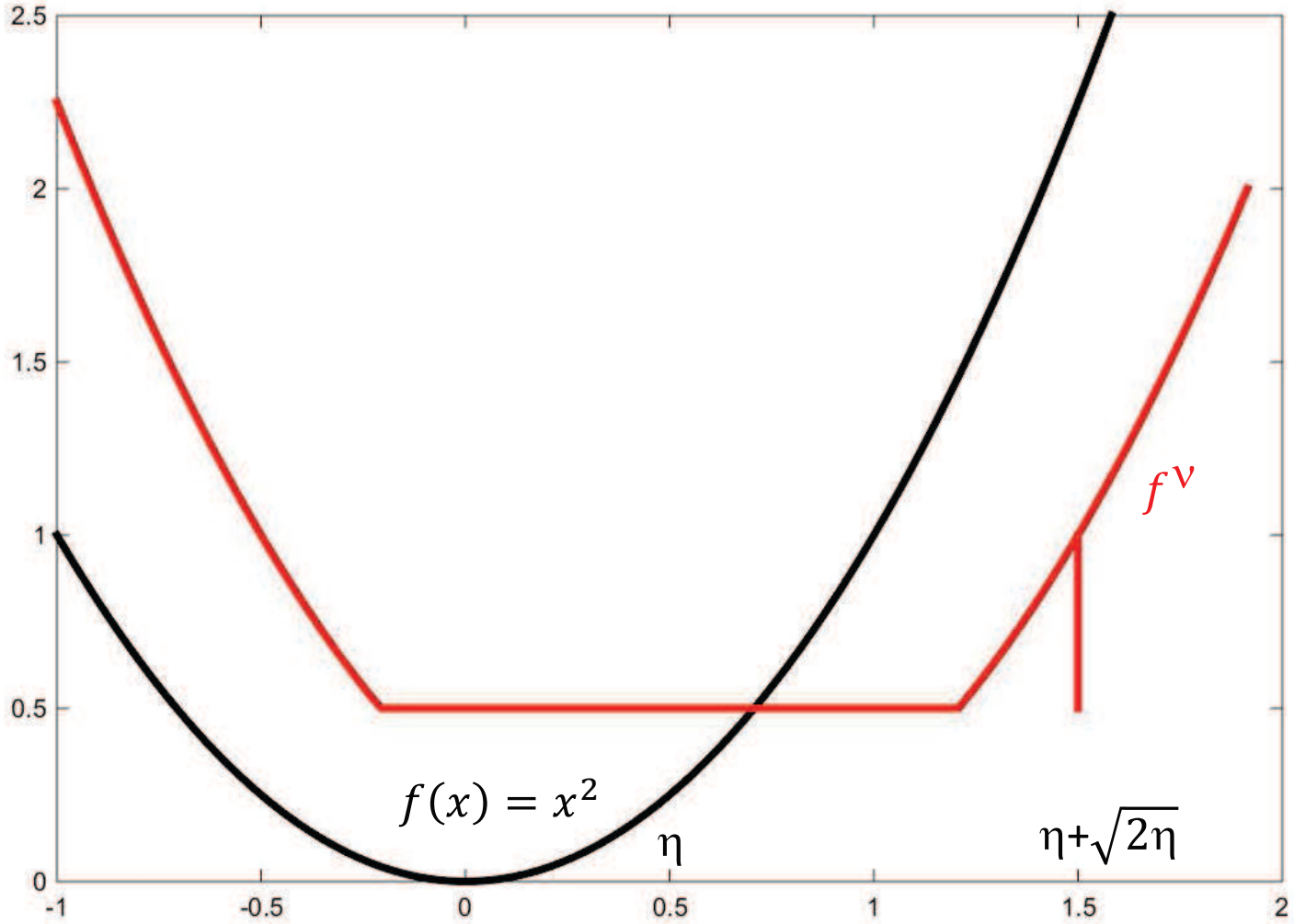
When $f(x) - \inf f \geq g(\operatorname{dist}(x, \operatorname{argmin} f))\ \forall x \in X$ for incr g

$$\operatorname{exs}\left(\operatorname{argmin} f^{\nu} \cap \mathbb{B}_X(\rho), \operatorname{argmin} f\right) \leq d\hat{l}_{\rho}(\operatorname{epi} f, \operatorname{epi} f^{\nu}) + g^{-1}\left(2d\hat{l}_{\rho}(\operatorname{epi} f, \operatorname{epi} f^{\nu})\right)$$



Sharpness of bound on minimizers

$d\hat{l}_\rho(\text{epi}\, f, \text{epi}\, f^\nu) = \eta = 1/2$; f has growth $g(t) = t^2$



$$\text{exs}\left(\text{argmin}\, f^\nu \cap \mathbb{B}_X(\rho), \text{argmin}\, f\right) \leq \eta + g^{-1}(2\eta)$$

Computing distances for compositions

For κ-Lipschitz $f : Y \to \mathbb{R}$ and $F, G : X \to Y$,

$$\hat{dl}_\rho\big(\operatorname{epi}(f \circ F),\ \operatorname{epi}(f \circ G)\big) \leq \max\{1, \kappa\}\hat{dl}_{\bar{\rho}}(\operatorname{gph} F, \operatorname{gph} G)$$

provided that $\bar{\rho}$ large enough

Distances for sums

$f_i, g_i : X \to [-\infty, \infty]$, $i = 1, 2$,
f_1, g_1 are Lipschitz continuous with common modulus κ

$$\hat{dl}_\rho\big(\operatorname{epi}(f_1 + f_2), \operatorname{epi}(g_1 + g_2)\big) \leq \sup_{A_\rho} |f_1 - g_1| + \big(1 + \kappa\big)\, \hat{dl}_{\bar{\rho}}(\operatorname{epi} f_2, \operatorname{epi} g_2)$$

provided that $\operatorname{epi}(f_1 + f_2)$ and $\operatorname{epi}(g_1 + g_2)$ are nonempty,
$A_\rho = \{f_1 + f_2 \leq \rho\} \cup \{g_1 + g_2 \leq \rho\} \cap \mathbb{B}_X(\rho)$,
$\bar{\rho} \geq \rho + \max\{0, -\inf_{\mathbb{B}_X(\rho)} f_1, -\inf_{\mathbb{B}_X(\rho)} g_1\}$

Convergence of stationary points

First-order conditions for $\text{minimize}_{x \in X} f(x)$:

$$\begin{aligned} \text{Oresme Rule:} \quad & df(x; w) \geq 0 \ \forall w \in X \\ \text{Fermat Rule:} \quad & 0 \in \partial f(x) \end{aligned}$$

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More generally:

For set-valued mapping $S : X \rightrightarrows Y$ and point $y^\star \in Y$
Generalized equation $y^\star \in S(x)$ has solution set $S^{-1}(y^\star)$

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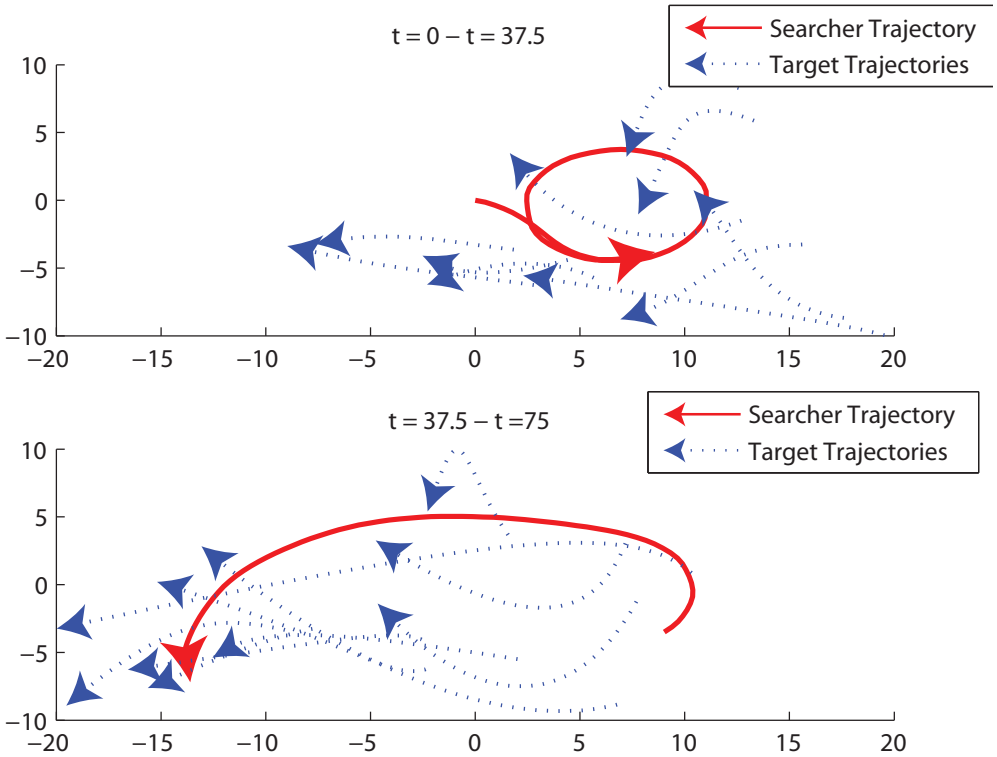
If $\operatorname{gph} S^\nu$ set-conv to $\operatorname{gph} S$, $y^\nu \to y^\star$, and $x^\nu \in (S^\nu)^{-1}(y^\nu) \to x^\star$, then $x^\star \in S^{-1}(y^\star)$

Convergence for Oresme Rule



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Sampling: Convergence of Oresme stationary points

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Solutions of generalized equations

For $\varepsilon \geq 0$, the **set of ε-solutions** is defined as

$$S^{-1}\big(\mathbb{B}_Y(y^\star, \varepsilon)\big) = \bigcup_{y \in \mathbb{B}_Y(y^\star, \varepsilon)} S^{-1}(y)$$

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Example: Optimality conditions for minimizing f over C

$$0 \in \partial f(x) + N_C(x)$$

With $S = \partial f + N_C$ and $y^\star = 0$, the set of ε-solutions becomes

$$S^{-1}\big(\mathbb{B}_{\mathbb{R}^n}(\varepsilon)\big) = \big\{x \in \mathbb{R}^n \mid 0 \in \partial f(x) + N_C(x) + \mathbb{B}_{\mathbb{R}^n}(\varepsilon)\big\}$$

Solution estimates for generalized equations

For metric spaces X and Y, suppose that $S, T : X \rightrightarrows Y$ have nonempty graphs, $0 \leq \varepsilon \leq \rho < \infty$, and $y^\star \in \mathbb{B}_Y(\rho - \varepsilon)$

Then,

$$\operatorname{exs}\Big(S^{-1}\big(\mathbb{B}_Y(y^\star, \varepsilon)\big) \cap \mathbb{B}_X(\rho);\ T^{-1}\big(\mathbb{B}_Y(y^\star, \delta)\big)\Big) \leq \widehat{dl}_\rho(\operatorname{gph} S, \operatorname{gph} T)$$

provided that $\delta > \varepsilon + \widehat{dl}_\rho(\operatorname{gph} S, \operatorname{gph} T)$

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provided that $\delta > \varepsilon + \hat{dl}_\rho(\operatorname{gph} S, \operatorname{gph} T)$

If X and Y have compact balls and $\operatorname{gph} T$ is closed, then the result also holds for $\delta = \varepsilon + \hat{dl}_\rho(\operatorname{gph} S, \operatorname{gph} T)$

Example: KKT solutions

$$\text{minimize } f_0(x) \text{ subject to } f_i(x) \leq 0 \text{ for } i = 1, \ldots, m \text{ (smooth)}$$

$(x, y) \in \mathbb{R}^{n+m}$ KKT solution if and only if $0 \in S(x, y)$

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where $S : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{3m+n}$ has

$$S(x, y) = \begin{pmatrix} [f_1(x), \infty) \\ \vdots \\ [f_m(x), \infty) \\ (-\infty, y_1] \\ \vdots \\ (-\infty, y_m] \\ \{y_1 f_1(x)\} \\ \vdots \\ \{y_m f_m(x)\} \\ \{\nabla f_0(x) + \sum_{i=1}^m y_i \nabla f_i(x)\} \end{pmatrix}$$

Estimates of KKT solutions

Let $g_0, \ldots, g_m$ define $T : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^{3m+n}$ similarly to S

Then,

$$\hat{dl}_\rho(\text{gph } S, \text{gph } T) \leq \max\left\{\delta, \rho\delta, (1 + m\rho)\eta\right\},$$

where

$$\delta = \max_{i=0,\ldots,m} \sup_{\|x\|_\infty \leq \rho} |f_i(x) - g_i(x)|$$

$$\eta = \max_{i=0,\ldots,m} \sup_{\|x\|_\infty \leq \rho} \|\nabla f_i(x) - \nabla g_i(x)\|_\infty$$

KKT system is stable while optimization problem may not be

References

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http://faculty.nps.edu/joroyset