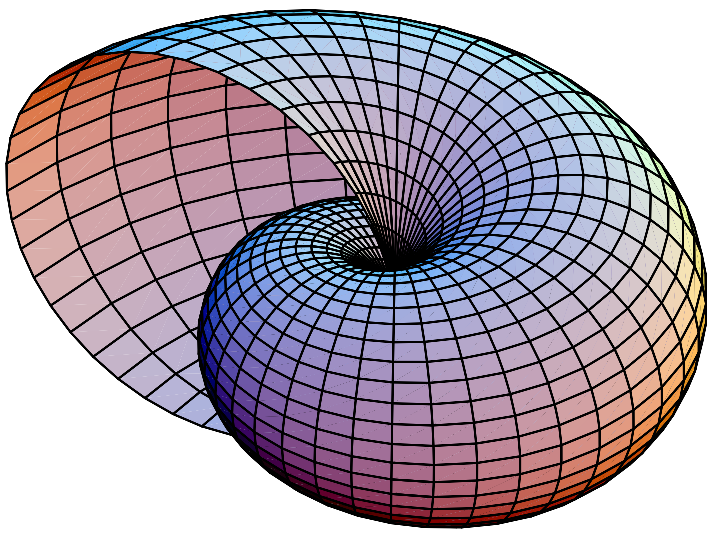
# MATHEMATICS TABLES



**Department of Applied Mathematics**
**Naval Postgraduate School**

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| *Initial typesetting:* | Carroll Wilde |
| *Graphics:* | David Canright |
| *Editing:* | Elle Zimmerman, David Canright |


Cover: in spherical coordinates $(\rho,\ \phi,\ \theta)$, shell surface is $\rho(\phi,\ \theta) = e^{k\theta} \sin \phi$, where $k = \frac{1}{\pi} \ln G$ and $G = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

## FOREWORD

This booklet provides convenient access to formulas and other data that are frequently used in mathematics courses. If a more comprehensive reference is needed, see, for example, the STANDARD MATHEMATICAL TABLES published by the Chemical Rubber Company, Cleveland, Ohio.

## DIFFERENTIALS AND DERIVATIVES

A. Letters $u$ and $v$ denote independent variables or functions of an independent variable; letters $a$ and $n$ denote constants.

B. To obtain a derivative, divide both members of the given formula for the differential by $du$ or by $dx$.

1. $d(a) = 0$
2. $d(au) = a\,du$
3. $d(u+v) = du + dv$
4. $d(uv) = u\,dv + v\,du$
5. $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$
6. $d(u^n) = n\,u^{n-1}\,du$
7. $d(e^u) = e^u du$
8. $d(a^u) = a^u \ln a\,du$
9. $d(\ln u) = \frac{du}{u}$
10. $d(\log_a u) = \frac{du}{u \ln a}$
11. $d(\sin u) = \cos u\,du$
12. $d(\cos u) = -\sin u\,du$
13. $d(\tan u) = \sec^2 u\,du$
14. $d(\cot u) = -\csc^2 u\,du$
15. $d(\sec u) = \sec u \tan u\,du$
16. $d(\csc u) = -\csc u \cot u\,du$
17. $d(\arcsin u) = \frac{du}{\sqrt{1-u^2}}$
18. $d(\arccos u) = -\frac{du}{\sqrt{1-u^2}}$
19. $d(\arctan u) = \frac{du}{1+u^2}$
20. $d(\sinh u) = \cosh u\,du$
21. $d(\cosh u) = \sinh u\,du$
22. $d(\text{arcsinh}\, u) = \frac{du}{\sqrt{u^2+1}}$
23. $d(\text{arccosh}\, u) = \frac{du}{\sqrt{u^2-1}}, \quad u > 1$
24. (Differentiation of Integrals) If $f$ is continuous, then
$$d\left(\int_a^u f(t)\,dt\right) = f(u)\,du.$$
25. (Chain Rule) If $y = f(u)$ and $u = g(x)$, then
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

## INTEGRALS

A. Letters $u$ and $v$ denote functions of an independent variable such as $x$; letters $a$, $b$, $m$, and $n$ denote constants.

B. Generally, each formula gives only one antiderivative. To find an expression for all antiderivatives, add a constant of integration.

## ELEMENTARY FORMS

1. $\int a\,du = a\,u$
2. $\int a\,f(u)\,du = a\int f(u)\,du$
3. $\int (f(u)+g(u))\,du = \int f(u)\,du + \int g(u)\,du$
4. $\int u\,dv = u\,v - \int v\,du$ (Integration by parts)
5. $\int u^n\,du = \dfrac{u^{n+1}}{n+1}, \qquad n \neq -1$
6. $\int \dfrac{du}{u} = \ln|u|$
7. $\int e^u du = e^u$
8. $\int a^u du = \dfrac{a^u}{\ln a}, \qquad a > 0$
9. $\int \cos u\,du = \sin u$
10. $\int \sin u\,du = -\cos u$
11. $\int \sec^2 u\,du = \tan u$
12. $\int \csc^2 u\,du = -\cot u$
13. $\int \sec u \tan u\,du = \sec u$
14. $\int \csc u \cot u\,du = -\csc u$

15. $\int \tan u\, du = -\ln|\cos u|$

16. $\int \cot u\, du = \ln|\sin u|$

17. $\int \sec u\, du = \ln|\sec u + \tan u|$

18. $\int \csc u\, du = \ln|\csc u - \cot u|$

19. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a}$

20. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a}$

21. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{u}{a}$

## INTEGRALS INVOLVING $au + b$

22. $\int (au + b)^n\, du = \frac{1}{a} \frac{(au + b)^{n+1}}{n + 1}, \qquad n \neq -1$

23. $\int \frac{du}{au + b} = \frac{1}{a} \ln|au + b|$

24. $\int \frac{u\, du}{au + b} = \frac{u}{a} - \frac{b}{a^2} \ln|au + b|$

25. $\int \frac{u\, du}{(au + b)^2} = \frac{b}{a^2(au + b)} + \frac{1}{a^2} \ln|au + b|$

26. $\int \frac{du}{u(au + b)} = \frac{1}{b} \ln\left|\frac{u}{au + b}\right|$

27. $\int u\sqrt{au + b}\, du = \frac{2(3au - 2b)}{15a^2}(au + b)^{\frac{3}{2}}$

28. $\int \frac{u\, du}{\sqrt{au + b}} = \frac{2(au - 2b)}{3a^2}\sqrt{au + b}$

29. $\int u^2\sqrt{au + b}\, du = \frac{2(8b^2 - 12abu + 15a^2u^2)}{105a^3}\,(au + b)^{\frac{3}{2}}$

30. $\int \frac{u^2\, du}{\sqrt{au + b}} = \frac{2(8b^2 - 4abu + 3a^2u^2)}{15a^3}\sqrt{au + b}$

## INTEGRALS INVOLVING $u^2 \pm a^2$

31. $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a}$ [See 19]

32. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|$

33. $\displaystyle\int \frac{u\,du}{u^2\pm a^2} = \frac{1}{2}\ln|u^2\pm a^2|$

34. $\displaystyle\int \frac{u^2\,du}{u^2-a^2} = u + \frac{a}{2}\ln\left|\frac{u-a}{u+a}\right|$

35. $\displaystyle\int \frac{u^2\,du}{u^2+a^2} = u - a\arctan\frac{u}{a}$

36. $\displaystyle\int \frac{du}{u(u^2\pm a^2)} = \pm\frac{1}{2a^2}\ln\left|\frac{u^2}{u^2\pm a^2}\right|$

37. $\displaystyle\int \frac{u\,du}{(u^2\pm a^2)^{n+1}} = -\frac{1}{2n(u^2\pm a^2)^n},\qquad n\neq 0$

## INTEGRALS INVOLVING $\sqrt{u^2\pm a^2}\,,\quad a>0$

38. $\displaystyle\int \frac{u\,du}{\sqrt{u^2\pm a^2}} = \sqrt{u^2\pm a^2}$ [See 5]

39. $\displaystyle\int u\sqrt{u^2\pm a^2}\,du = \frac{1}{3}(u^2\pm a^2)^{\frac{3}{2}}$ [See 5]

40. $\displaystyle\int \frac{du}{\sqrt{u^2\pm a^2}} = \ln|u+\sqrt{u^2\pm a^2}|$

41. $\displaystyle\int \sqrt{u^2\pm a^2}\,du = \frac{u}{2}\sqrt{u^2\pm a^2} \pm \frac{a^2}{2}\ln|u+\sqrt{u^2\pm a^2}|$

42. $\displaystyle\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\ln\left|\frac{u}{a+\sqrt{u^2+a^2}}\right|$

43. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\text{arcsec}\frac{u}{a}$ [See 21]

44. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\ \text{arcsec}\frac{u}{a}$

45. $\displaystyle\int \frac{\sqrt{u^2+a^2}}{u}\,du = \sqrt{u^2+a^2} + a\ln\left|\frac{u}{a+\sqrt{u^2+a^2}}\right|$

## INTEGRALS INVOLVING $\sqrt{u^2 \pm a^2}$, $a > 0$ (Continued)

46. $$\int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}|$$

47. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{4}(u^2 \pm a^2)^{\frac{3}{2}} \mp \frac{a^2 u}{8}\sqrt{u^2 \pm a^2} - \frac{a^4}{8}\ln|u + \sqrt{u^2 \pm a^2}|$$

## INTEGRALS INVOLVING $\sqrt{a^2 - u^2}$, $a > 0$

48. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} \qquad \text{[See 20]}$$

49. $$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}$$

50. $$\int \frac{u\,du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2} \qquad \text{[See 5]}$$

51. $$\int u\sqrt{a^2 - u^2}\,du = -\frac{1}{3}(a^2 - u^2)^{\frac{3}{2}} \qquad \text{[See 5]}$$

52. $$\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} + \ln\left|\frac{u}{a + \sqrt{a^2 - u^2}}\right|$$

53. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\ln\left|\frac{u}{a + \sqrt{a^2 - u^2}}\right|$$

54. $$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}$$

55. $$\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin\frac{u}{a}$$

56. $$\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a}$$

57. $$\int u^2\sqrt{a^2 - u^2}\,du = -\frac{u}{4}(a^2 - u^2)^{3/2} + \frac{a^2 u}{8}\sqrt{a^2 - u^2} + \frac{a^4}{8}\arcsin\frac{u}{a}$$

## INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

58. $$\int \sin^2 au\, du = \frac{u}{2} - \frac{\sin 2au}{4a}$$

59. $$\int \cos^2 au\, du = \frac{u}{2} + \frac{\sin 2au}{4a}$$

60. $$\int \sin^3 au\, du = \frac{\cos^3 au}{3a} - \frac{\cos au}{a}$$

61. $$\int \cos^3 au\, du = \frac{\sin au}{a} - \frac{\sin^3 au}{3a}$$

62. $$\int \sin^n au\, du = -\frac{\sin^{n-1} au \cos au}{na} + \frac{n-1}{n} \int \sin^{n-2} au\, du,$$

63. $$\int \cos^n au\, du = \frac{\cos^{n-1} au \sin au}{na} + \frac{n-1}{n} \int \cos^{n-2} au\, du,$$

64. $$\int \sin^2 au \cos^2 au\, du = \frac{u}{8} - \frac{\sin 4au}{32a}$$

65. $$\int \frac{du}{\sin^2 au} = -\frac{1}{a} \cot au$$

66. $$\int \frac{du}{\cos^2 au} = \frac{1}{a} \tan au$$

67. $$\int \tan^2 au\, du = \frac{1}{a} \tan au - u$$

68. $$\int \cot^2 au\, du = -\frac{1}{a} \cot au - u$$

69. $$\int \sec^3 au\, du = \frac{1}{2a} \sec au \tan au + \frac{1}{2a} \ln |\sec au + \tan au|$$

70. $$\int \csc^3 au\, du = -\frac{1}{2a} \csc au \cot au + \frac{1}{2a} \ln |\csc au - \cot au|$$

71. $$\int u \sin au\, du = \frac{1}{a^2} \sin au - \frac{u}{a} \cos au$$

72. $$\int u \cos au\, du = \frac{1}{a^2} \cos au + \frac{u}{a} \sin au$$

73. $$\int u^2 \sin au\, du = \frac{2u}{a^2} \sin au - \frac{a^2u^2 - 2}{a^3} \cos au$$

74. $$\int u^2 \cos au\, du = \frac{2u}{a^2} \cos au + \frac{a^2u^2 - 2}{a^3} \sin au$$

## INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

75. $$\int ue^{au}\,du = \frac{e^{au}}{a^2}(au-1)$$

76. $$\int u^2e^{au}\,du = \frac{e^{au}}{a^3}(a^2u^2-2au+2)$$

77. $$\int u^ne^{au}\,du = \frac{u^ne^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}\,du, \quad n>1$$

78. $$\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu)$$

79. $$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)$$

80. $$\int \frac{du}{1+e^{au}} = u - \frac{1}{a}\ln(1+e^{au})$$

## MISCELLANEOUS INTEGRALS

81. $$\int \arcsin au\,du = u\arcsin au + \frac{\sqrt{1-a^2u^2}}{a}$$

82. $$\int \arccos au\,du = u\arccos au - \frac{\sqrt{1-a^2u^2}}{a}$$

83. $$\int \arctan au\,du = u\arctan au - \frac{1}{2a}\ln(1+a^2u^2)$$

84. $$\int \operatorname{arccot} au\,du = u\operatorname{arccot} au + \frac{1}{2a}\ln(1+a^2u^2)$$

85. $$\int \ln au\,du = u\ln au - u$$

86. $$\int u^n\ln au\,du = u^{n+1}\left[\frac{\ln au}{n+1} - \frac{1}{(n+1)^2}\right]$$

87. $$\int_0^\infty e^{-a^2u^2}\,du = \frac{\sqrt{\pi}}{2a}$$

## WALLIS' FORMULAS

88. $$\int_0^{\frac{\pi}{2}} \sin^m u\,du = \begin{cases} \frac{(m-1)(m-3)\cdots(3)(1)}{(m)(m-2)\cdots(4)(2)}\,\frac{\pi}{2}, & m \text{ even;} \\ \frac{(m-1)(m-3)\cdots(4)(2)}{(m)(m-2)\cdots(5)(3)}, & m \text{ odd} \end{cases}$$

89. $$\int_0^{\frac{\pi}{2}} \cos^m u\,du = \int_0^{\frac{\pi}{2}} \sin^m u\,du$$

90. $$\int_0^{\frac{\pi}{2}} \sin^m u \cos^n u\,du =$$
$$\begin{cases} \frac{(m-1)(m-3)\cdots(3)(1)(n-1)(n-3)\cdots(4)(2)}{(m+n)(m+n-2)\cdots(3)(1)}, & m \text{ even, } n \text{ odd;} \\ \frac{(m-1)(m-3)\cdots(4)(2)(n-1)(n-3)\cdots(3)(1)}{(m+n)(m+n-2)\cdots(3)(1)}, & m \text{ odd, } n \text{ even;} \\ \frac{(m-1)(m-3)\cdots(4)(2)(n-1)(n-3)\cdots(4)(2)}{(m+n)(m+n-2)\cdots(4)(2)}, & m \text{ odd, } n \text{ odd;} \\ \frac{(m-1)(m-3)\cdots(3)(1)(n-1)(n-3)\cdots(3)(1)}{(m+n)(m+n-2)\cdots(4)(2)}\,\frac{\pi}{2}, & m \text{ even, } n \text{ even} \end{cases}$$

## GAMMA FUNCTION

Definition :

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt, \qquad x > 0$$

Shift Property :

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0$$

Definition :

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \qquad x < 0,\ x \neq -1, -2, \ldots$$

Special Values :

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(n) = (n-1)!, \qquad n = 1, 2, \ldots$$

## LAPLACE TRANSFORMS

| | | |
|---|---|---|
| 1. | $f(t)$ | $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ |
| 2. | $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| 3. | $f'(t)$ | $sF(s) - f(0)$ |
| 4. | $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ |
| 5. | $t^n f(t), \quad n = 1, 2, 3, \ldots$ | $(-1)^n F^{(n)}(s)$ |
| 6. | $e^{at} f(t)$ | $F(s-a)$ |
| 7. | $f(t+P) \equiv f(t)$ | $\frac{\int_0^P e^{-st} f(t)\,dt}{1-e^{-sP}}$ |
| 8. | $f(t)H(t-a)$ | $e^{-as} L\{f(t+a)\}$ |
| 9. | $f(t-a)H(t-a)$ | $e^{-as} F(s)$ |
| 10. | $\int_0^t f(u)\,du$ | $\frac{F(s)}{s}$ |
| 11. | $\int_0^t f(u)g(t-u)\,du$ | $F(s)G(s)$ |
| 12. | $\frac{f(t)}{t}$ | $\int_s^\infty F(v)\,dv$ |
| 13. | $1,\ H(t)^*,\ u(t)$ | $\frac{1}{s}$ |
| 14. | $t^n, \quad n = 1, 2, 3, \ldots$ | $\frac{n!}{s^{n+1}}$ |
| 15. | $t^a, \quad a > -1$ | $\frac{\Gamma(a+1)}{s^{a+1}}$ |
| 16. | $e^{at}$ | $\frac{1}{s-a}$ |
| 17. | $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| 18. | $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| 19. | $\frac{1}{a}(e^{at} - 1)$ | $\frac{1}{s(s-a)}$ |
| 20. | $\frac{1}{a-b}(e^{at} - e^{bt})$ | $\frac{1}{(s-a)(s-b)}, \quad a \neq b$ |
| 21. | $te^{at}$ | $\frac{1}{(s-a)^2}$ |
| 22. | $t^n e^{at}, \quad n = 1, 2, 3, \ldots$ | $\frac{n!}{(s-a)^{n+1}}$ |
| 23. | $\sinh at$ | $\frac{a}{s^2-a^2}$ |
| 24. | $\cosh at$ | $\frac{s}{s^2-a^2}$ |
| 25. | $\frac{1}{a^2}(1 - \cos at)$ | $\frac{1}{s(s^2+a^2)}$ |
| 26. | $\frac{1}{a^3}(at - \sin at)$ | $\frac{1}{s^2(s^2+a^2)}$ |

* The Heaviside step function $H$ is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0. \end{cases}$$

## LAPLACE TRANSFORMS (Continued)

| | | |
|---|---|---|
| 27. | $\frac{1}{2a^3}(\sin at - at\cos at)$ | $\frac{1}{(s^2+a^2)^2}$ |
| 28. | $\frac{1}{2a}t\sin at$ | $\frac{s}{(s^2+a^2)^2}$ |
| 29. | $\frac{1}{2a}(\sin at + at\cos at)$ | $\frac{s^2}{(s^2+a^2)^2}$ |
| 30. | $t\cos at$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| 31. | $\frac{b\sin at - a\sin bt}{ab(b^2-a^2)}$ | $\frac{1}{(s^2+a^2)(s^2+b^2)}, \quad a \neq b$ |
| 32. | $\frac{\cos at - \cos bt}{(b^2-a^2)}$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}, \quad a \neq b$ |
| 33. | $\frac{a\sin at - b\sin bt}{(a^2-b^2)}$ | $\frac{s^2}{(s^2+a^2)(s^2+b^2)}, \quad a \neq b$ |
| 34. | $e^{-bt}\sin\omega t$ | $\frac{\omega}{(s+b)^2+\omega^2}$ |
| 35. | $e^{-bt}\cos\omega t$ | $\frac{s+b}{(s+b)^2+\omega^2}$ |
| 36. | $1-\cos\omega t$ | $\frac{\omega^2}{s(s^2+\omega^2)}$ |
| 37. | $\delta(t-a)$ | $e^{-sa}$ |
| 38. | $\frac{\sqrt{(a-b)^2+\omega^2}}{\omega}e^{-bt}\sin(\omega t+\Psi)$ where $\Psi = \arctan\frac{\omega}{a-b}$ | $\frac{s+a}{(s+b)^2+\omega^2}$ |
| 39. | $1+\frac{\sqrt{b^2+\omega^2}}{\omega}e^{-bt}\sin(\omega t-\Psi)$ where $\Psi = \arctan\frac{\omega}{-b}$ | $\frac{b^2+\omega^2}{s[(s+b)^2+\omega^2]}$ |
| 40. | $1+\frac{e^{-act}}{\sqrt{1-c^2}}\sin(\sqrt{1-c^2}at-\Psi)$ where $\Psi = \arctan\frac{\sqrt{1-c^2}}{-c}, \quad -1 < c < 1$ | $\frac{a^2}{s[s^2+2acs+a^2]}$ |
| 41. | $\frac{a}{c^2}+\frac{\sqrt{(a-b)^2+\omega^2}}{c\omega}e^{-bt}\sin(\omega t-\Psi)$ where $\Psi = \arctan\frac{a\omega}{c^2-ab}, \quad c^2 = b^2+\omega^2$ | $\frac{s+a}{s[(s+b)^2+\omega^2]}$ |

## PROBABILITY AND STATISTICS

Formulas for counting permutations and combinations:

(a) $_nP_k = \dfrac{n!}{(n-k)!}$ (b) $_nC_k = \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Let $A$ and $B$ be subsets (events) of the same sample space.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) The <u>conditional probability of $B$ given $A$, $P(B|A)$</u>, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \qquad P(A) \neq 0.$$

(c) $A$ and $B$ are <u>statistically independent</u> if and only if

$$P(A \cap B) = P(A)P(B).$$

Let $X$ be a discrete random variable with probability function $f(x)$.

(a) $P(X = x) = f(x), \qquad \text{where } x \in S \equiv \text{ range of } X$

(b) $E(X) = \sum_S x f(x) = \mu = \underline{\text{ mean of } X}$

(c) $\text{var}(X) = E((X-\mu)^2) = E(X^2) - \mu^2 = \sum_S x^2 f(x) - \mu^2 = \sigma^2$

$= \underline{\text{variance of } X}$

(d) $\sigma = \sqrt{\text{var}(X)} = \underline{\text{standard deviation of } X}$.

Let $X$ be a continuous random variable with and probability density $f(x)$.

(a) $P(a \le X \le b) = \int_a^b f(x)\,dx$.

(b) $E(X) = \int_S x f(x)\,dx = \mu = \underline{\text{ mean of } X}$

(c) $var(X) = E((X-\mu)^2) = E(X^2) - \mu^2 = \int_S x^2 f(x)\,dx - \mu^2 = \sigma^2$

$= \underline{\text{variance of } X}$

(d) $\sigma = \sqrt{\text{var}(X)} = \underline{\text{standard deviation of } X}$.

Let $X_1, X_2, \ldots, X_n$ be $n$ statistically independent random variables, each having the same expected value, $\mu$, and the same variance, $\sigma^2$. Let

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Then $Y$ is a random variable for which $E(Y) = \mu$ and $\text{var}(Y) = \frac{\sigma^2}{n}$.

## PROBABILITY FUNCTIONS (Discrete Random Variables)

Binomial

$$f(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad (x = 0,1,2,\ldots,n),$$

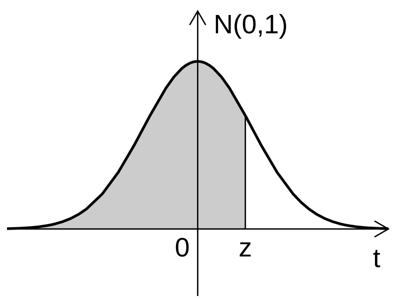
$$\mu = np, \qquad \sigma^2 = np(1-p).$$

Poisson

$$f(x;a) = \frac{e^{-a}a^x}{x!}, \qquad (x = 0,1,2,\ldots),$$

$$\mu = \sigma^2 = a.$$

## STANDARD NORMAL CUMULATIVE DISTRIBUTION FUNCTION




$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}}\, dt$$

## TABLE OF NORMAL AREAS

| $z$ | $F(z)$ | $z$ | $F(z)$ | $z$ | $F(z)$ | $z$ | $F(z)$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.500 | 0.8 | 0.788 | 1.6 | 0.945 | 2.4 | 0.992 |
| 0.1 | 0.540 | 0.9 | 0.816 | 1.7 | 0.955 | 2.5 | 0.994 |
| 0.2 | 0.579 | 1.0 | 0.841 | 1.8 | 0.964 | 2.6 | 0.995 |
| 0.3 | 0.618 | 1.1 | 0.864 | 1.9 | 0.971 | 2.7 | 0.997 |
| 0.4 | 0.655 | 1.2 | 0.885 | 2.0 | 0.977 | 2.8 | 0.997 |
| 0.5 | 0.691 | 1.3 | 0.903 | 2.1 | 0.982 | 2.9 | 0.998 |
| 0.6 | 0.726 | 1.4 | 0.919 | 2.2 | 0.986 | 3.0 | 0.999 |
| 0.7 | 0.758 | 1.5 | 0.933 | 2.3 | 0.989 | | |

## PROBABILITY FUNCTIONS (Continuous Random Variables)

Uniform

$$f(x;a,b) = \frac{1}{b-a} \qquad (a \le x \le b).$$

$$\mu = \frac{a+b}{2}; \qquad \sigma^2 = \frac{(b-a)^2}{12}.$$

Exponential

$$f(x;b) = be^{-bx} \qquad (x \ge 0,\ b > 0).$$

$$\mu = \frac{1}{b}; \qquad \sigma^2 = \frac{1}{b^2}.$$

Normal

$$f(x;a,b) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}} \qquad (-\infty < x < \infty).$$

$$\mu = a; \qquad \sigma^2 = b^2.$$

## FOURIER SERIES

The Fourier series expansion of a function $f(t)$ is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt, \quad n \ge 1$$

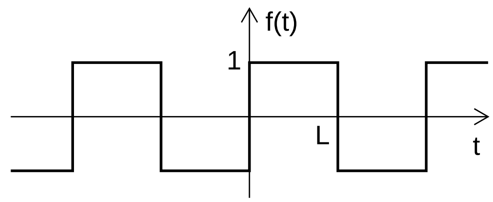
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt, \quad n \ge 1$$

(Note: for the complex Fourier Series, see page 29)

## FOURIER SERIES (Continued)

Examples of Fourier Series

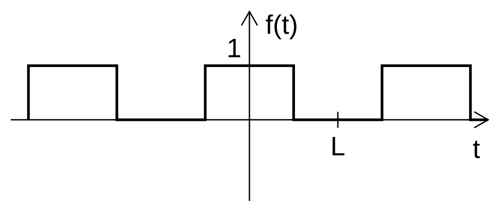
$$f(t) = \begin{cases} 1, & 0 < t < L; \\ -1, & -L < t < 0. \end{cases}$$




$$\frac{4}{\pi}\left(\sin\frac{\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} + \frac{1}{5}\sin\frac{5\pi t}{L} + \cdots\right)$$

---

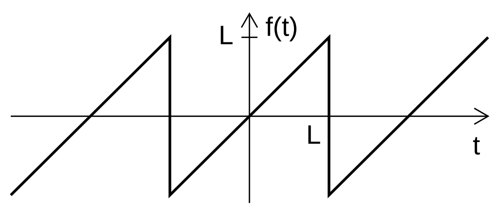
$$f(t) = \begin{cases} 0, & \frac{L}{2} < |t| < L \\ 1, & 0 < |t| < \frac{L}{2}. \end{cases}$$




$$\frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi t}{L} - \frac{1}{3}\cos\frac{3\pi t}{L} + \frac{1}{5}\cos\frac{5\pi t}{L} - \ + \cdots\right)$$

---

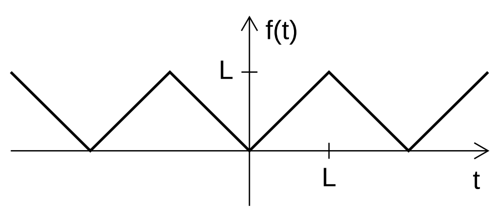
$$f(t) = t, \qquad -L < t < L$$




$$\frac{2L}{\pi}\left(\sin\frac{\pi t}{L} - \frac{1}{2}\sin\frac{2\pi t}{L} + \frac{1}{3}\sin\frac{3\pi t}{L} - \ + \cdots\right)$$

---

$$f(t) = |t|, \qquad -L < t < L$$




$$\frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi t}{L} + \frac{1}{3^2}\cos\frac{3\pi t}{L} + \frac{1}{5^2}\cos\frac{5\pi t}{L} + \cdots\right)$$

## SEPARATION OF VARIABLES

Eigenvalues and Eigenfunctions for the differential equation

$$\varphi'' + \lambda\varphi = 0.$$

1. $\varphi(0) = \varphi(L) = 0$

   $\lambda_n = (\frac{n\pi}{L})^2, \quad \varphi_n = \sin\frac{n\pi}{L}x, \quad n = 1,\ 2,\ \ldots$

2. $\varphi'(0) = \varphi(L) = 0$

   $\lambda_n = \left[(n-\frac{1}{2})\frac{\pi}{L}\right]^2, \quad \varphi_n = \cos(n-\frac{1}{2})\frac{\pi}{L}x, \quad n = 1,\ 2,\ \ldots$

3. $\varphi(0) = \varphi'(L) = 0$

   $\lambda_n = \left[(n-\frac{1}{2})\frac{\pi}{L}\right]^2, \quad \varphi_n = \sin(n-\frac{1}{2})\frac{\pi}{L}x, \quad n = 1,\ 2,\ \ldots$

4. $\varphi'(0) = \varphi'(L) = 0$

   $\lambda_n = (\frac{n\pi}{L})^2, \quad \varphi_n = \cos\frac{n\pi}{L}x, \quad n = 0,\ 1,\ 2,\ \ldots$

5. $\varphi(-L) = \varphi(L), \quad \varphi'(-L) = \varphi'(L)$

   $\lambda_0 = 0, \quad \varphi_0 = 1,$

   $\lambda_n = (\frac{n\pi}{L})^2, \quad \varphi_n = \sin\frac{n\pi}{L}x \quad \text{and} \quad \cos\frac{n\pi}{L}x, \quad n = 1,\ 2,\ \ldots$

Solution of inhomogeneous ordinary differential equations

1. $u_n' + k\,\lambda_n u_n = h_n(t)$

   $u_n(t) = a_n e^{-k\lambda_n t} + \int_0^t h_n(\tau) e^{-k\lambda_n(t-\tau)}\,d\tau$

2. $u_n'' + \mu_n^2\,u_n = h_n(t)$

   $u_n(t) = a_n \cos\mu_n t + b_n \sin\mu_n t + \frac{1}{\mu_n}\int_0^t h_n(\tau)\sin\mu_n(t-\tau)\,d\tau$

## BESSEL FUNCTIONS

1. The differential equation

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0$$

has solutions

$$y = y_n(\lambda x) = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x),$$

where

$$J_n(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{n+2m}}{2^{n+2m} m! \Gamma(n+m+1)}.$$

2. General properties:

$J_{-n}(t) = (-1)^n J_n(t);\qquad J_0(0) = 1;\qquad J_n(0) = 0,\qquad n = 1, 2, 3, \ldots;$

for fixed $n$, $J_n(t) = 0$ has infinitely many solutions.

3. Identities:

(a) $\frac{d}{dt}[t^n J_n(t)] = t^n J_{n-1}(t)$.

(b) $\frac{d}{dt}[t^{-n} J_n(t)] = -t^{-n} J_{n+1}(t);\qquad J_0'(t) = -J_1(t)$

(c) $J_n'(t) = \frac{1}{2}[J_{n-1}(t) - J_{n+1}(t)]$

$\qquad = J_{n-1}(t) - \frac{n}{t} J_n(t) = \frac{n}{t} J_n(t) - J_{n+1}(t)$

(d) $J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t)$ (recurrence).

4. Orthogonality:

Solutions $y_n(\lambda_0 x), y_n(\lambda_1 x), \ldots, y_n(\lambda_i x), \ldots$ of the differential eigensystem

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0$$

$$A_1 y(x_1) - B_1 y'(x_1) = 0,\qquad A_2 y(x_2) + B_2 y'(x_2) = 0$$

have the property

$$\int_{x_1}^{x_2} x y_n(\lambda_i x) y_n(\lambda_j x)\, dx = 0$$

for $i \neq j$, and

$$\int_{x_1}^{x_2} x y_n^2(\lambda_i x)\, dx = \left. \frac{(\lambda_i^2 x^2 - n^2) y_n^2(\lambda_i x) + x^2 \left[\frac{d}{dx} y_n(\lambda_i x)\right]^2}{2\lambda_i^2} \right|_{x_1}^{x_2}$$

for $i = j$.

5. Integration properties:

$$(a) \quad \int t^{\nu} J_{\nu-1}(t)\,dt = t^{\nu} J_{\nu}(t) + C$$

$$(b) \quad \int t^{-\nu} J_{\nu+1}(t)\,dt = -t^{-\nu} J_{\nu}(t) + C$$

$$(c) \quad \int t J_0(t)\,dt = t J_1(t) + C$$

$$(d) \quad \int t^3 J_0(t)\,dt = (t^3 - 4t) J_1(t) + 2t^2 J_0(t) + C$$

$$(e) \quad \int t^2 J_1(t)\,dt = 2t J_1(t) - t^2 J_0(t) + C$$

$$(f) \quad \int t^4 J_1(t)\,dt = (4t^3 - 16t) J_1(t) - (t^4 - 8t^2) J_0(t) + C$$

## LEGENDRE POLYNOMIALS

The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

has solution $y = P_n(x)$, where

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m}$$

on the interval $[-1, 1]$. The Legendre polynomials can also be obtained iteratively from the first two,

$$P_0(x) = 1 \qquad \text{and} \qquad P_1(x) = x,$$

and the recurrence relation,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

The Legendre polynomials are also given by Rodrigues' formula:

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x^2)^n].$$

Orthogonality:

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0, & n \neq m; \\ \frac{2}{2n+1}, & n = m. \end{cases}$$

Standardization:

$$P_n(1) = 1.$$

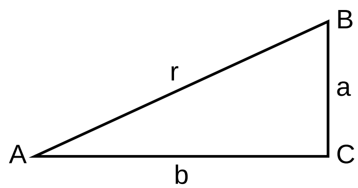
## TRIGONOMETRIC FUNCTIONS IN A RIGHT TRIANGLE

If $A$, $B$, and $C$ are the vertices ($C$ the right angle), and $a$, $b$, and $r$ the sides opposite, respectively, then

$$\text{sine } A = \sin A = \frac{a}{r}, \qquad \text{cosine } A = \cos A = \frac{b}{r},$$

$$\text{tangent } A = \tan A = \frac{a}{b}, \qquad \text{cotangent } A = \cot A = \frac{b}{a},$$

$$\text{secant } A = \sec A = \frac{r}{b}, \qquad \text{cosecant } A = \csc A = \frac{r}{a},$$




$$\text{exsecant } A = \text{exsec } A = \sec A - 1$$

$$\text{versine } A = \text{vers } A = 1 - \cos A$$

$$\text{coversine } A = \text{covers } A = 1 - \sin A$$

$$\text{haversine } A = \text{hav } A = \frac{1}{2}\text{vers } A$$

## TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

| | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| sin | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ | $-1$ |
| cos | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ | $0$ |
| tan | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | $\infty$ | $0$ | $\infty$ |
| cot | $\infty$ | $\sqrt{3}$ | $1$ | $\frac{\sqrt{3}}{3}$ | $0$ | $\infty$ | $0$ |
| sec | $1$ | $\frac{2\sqrt{3}}{3}$ | $\sqrt{2}$ | $2$ | $\infty$ | $-1$ | $\infty$ |
| csc | $\infty$ | $2$ | $\sqrt{2}$ | $\frac{2\sqrt{3}}{3}$ | $1$ | $\infty$ | $-1$ |

## COMPLEX EXPONENTIAL FORMS ($i^2 = -1$)

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \qquad \csc x = \frac{2i}{e^{ix} - e^{-ix}}$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \sec x = \frac{2}{e^{ix} + e^{-ix}}$$

$$\tan x = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \qquad \cot x = i\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$

## RELATIONS AMONG THE FUNCTIONS

$\sin x = \frac{1}{\csc x}$

$\cos x = \frac{1}{\sec x}$

$\tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$^* \sin x = \pm\sqrt{1 - \cos^2 x}$

$^* \tan x = \pm\sqrt{\sec^2 x - 1}$

$^* \cot x = \pm\sqrt{\csc^2 x - 1}$

$\sin x = \cos(\frac{\pi}{2} - x) = \sin(\pi - x)$

$\tan x = \cot(\frac{\pi}{2} - x) = -\tan(\pi - x)$

$\csc x = \frac{1}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$1 + \tan^2 x = \sec^2 x$

$^* \cos x = \pm\sqrt{1 - \sin^2 x}$

$^* \sec x = \pm\sqrt{1 + \tan^2 x}$

$^* \csc x = \pm\sqrt{1 + \cot^2 x}$

$\cos x = \sin(\frac{\pi}{2} - x) = -\cos(\pi - x)$

$\cot x = \tan(\frac{\pi}{2} - x) = -\cot(\pi - x)$

## SUMS AND MULTIPLES OF ANGLES

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin 3x = 3 \sin x - 4 \sin^3 x$$
$$\cos 3x = 4 \cos^3 x - 3 \cos x$$
$$\sin 4x = 8 \sin x \cos^3 x - 4 \sin x \cos x$$
$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$$
$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$
$$^* \sin \frac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{2}} \qquad ^* \cos \frac{1}{2}x = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$^* \tan \frac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

* The choice of the sign in front of the radical depends on the quadrant in which $x$, regarded as an angle, falls.

## MISCELLANEOUS RELATIONS

$\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$
$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$
$\cos x \sin y = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$
$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$
$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \ \cos y}$ $\qquad \cot x \pm \cot y = \frac{\pm \sin(x \pm y)}{\sin x \ \sin y}$
$\frac{1+\tan x}{1-\tan x} = \tan(\frac{\pi}{4} + x)$ $\qquad \frac{\cot x+1}{\cot x-1} = \cot(\frac{\pi}{4} - x)$
$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \frac{1}{2}(x \pm y)$ $\qquad \frac{\sin x \pm \sin y}{\cos x - \cos y} = -\cot \frac{1}{2}(x \mp y)$
$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\tan \frac{1}{2}(x+y)}{\tan \frac{1}{2}(x-y)}$
$\sin^2 x - \sin^2 y = \sin(x + y) \ \sin(x - y)$
$\cos^2 x - \cos^2 y = -\sin(x + y) \ \sin(x - y)$
$\cos^2 x - \sin^2 y = \cos(x + y) \ \cos(x - y)$

## INVERSE TRIGONOMETRIC FUNCTIONS

The following table gives the domains of the inverse trigonometric functions.

| Function | Interval containing principal value | |
|---|---|---|
| | x positive or zero | x negative |
| $y = \arcsin x$ and $y = \arctan x$ | $0 \leq y \leq \frac{\pi}{2}$ | $-\frac{\pi}{2} \leq y \leq 0$ |
| $y = \arccos x$ and $y = \text{arccot} x$ | $0 \leq y \leq \frac{\pi}{2}$ | $\frac{\pi}{2} \leq y \leq \pi$ |
| $y = \text{arcsec} x$ and $y = \text{arccsc} x$ | $0 \leq y \leq \frac{\pi}{2}$ | $-\pi \leq y \leq -\frac{\pi}{2}$ |

Two notation systems are in common use. For example, the inverse sine is often denoted by arcsin or $\sin^{-1}$. In many references, however, "principal value"is used for the function and is denoted by Arcsin or $\text{Sin}^{-1}$. Thus, in one book,

$$\text{Arc}\sin \frac{1}{2} = \text{Sin}^{-1} x = \frac{\pi}{6} \quad \text{and} \quad \arcsin x = \frac{\pi}{6} + 2n\pi,$$

while in another, only lower case is used:

$$\arcsin \frac{1}{2} = \frac{\pi}{6}.$$

Special care is advisable in the use of reference materials.

## SIDE-ANGLE RELATIONS IN PLANE TRIANGLES

Letters $A$, $B$, and $C$ denote the angles of a plane triangle with opposite sides $a$, $b$, and $c$, respectively. The letter $s$ denotes the semi-perimeter of the triangle, that is,

$$s = \frac{1}{2}(a+b+c).$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{ diameter of the circumscribed circle}$$

$$a^2 = b^2 + c^2 - 2\,b\,c\,\cos A$$

$$a = b\,\cos C + c\,\cos B$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

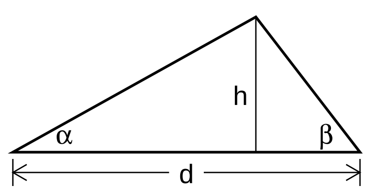
$$\text{area } = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

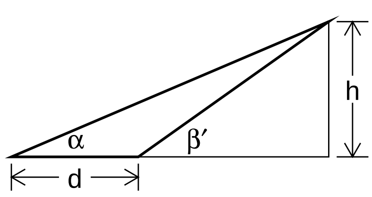
$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}$$




$$h = d\,\frac{\sin\alpha\,\sin\beta}{\sin(\alpha+\beta)} = \frac{d}{\cot\alpha + \cot\beta}$$




$$h = d\,\frac{\sin\alpha\,\sin\beta'}{\sin(\beta'-\alpha)} = \frac{d}{\cot\alpha - \cot\beta'}$$

## HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \pm\frac{\sinh x}{\cosh x} = \frac{1}{\coth x}$$

$$\sinh x = \pm\sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} = \pm\frac{1}{\sqrt{\coth^2 x - 1}}$$

$$\cosh x = \sqrt{1+\sinh^2 x} = \frac{1}{\sqrt{1-\tanh^2 x}} = \pm\frac{\coth x}{\sqrt{\coth^2 x - 1}}$$

$$\tanh x = \frac{\sinh x}{\sqrt{1+\sinh^2 x}} = \pm\frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \frac{1}{\coth x}$$

$$\coth x = \frac{\sqrt{1+\sinh^2 x}}{\sinh x} = \pm\frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \qquad \coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$$

$$\sinh 2x = 2\cosh x \sinh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \qquad \coth^{-1} x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$$

Complex hyperbolic functions involve the Euler relations

$$e^{iz} = \cos z \; + \; i\sin z; \qquad \cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

$$\cos z = \cosh iz \qquad \cosh z = \cos iz$$

$$\sin z = -i\sinh iz \qquad \sinh z = -i\sin iz$$

$$\tan z = -i\tanh iz \qquad \tanh z = -i\tan iz$$

$$\cot z = i\coth iz \qquad \coth z = i\cot iz$$

$$\sinh(x \pm iy) = \sinh x \cos y \pm i\cosh x \;\sin y$$

$$\cosh(x \pm iy) = \cosh x \cos y \pm i\sinh x \;\sin y$$

$$\ln z = \ln|z| + \; i\arg z; \quad \text{so} \quad \ln ix = \ln|x| + \; (2n+\frac{1}{2})\pi i$$

$$\text{and } \ln(-x) = \ln|x| + \; (2n+1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$$

## FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

I. Separable equations : $N(y)\,dy = M(x)\,dx$. To solve, integrate both sides:

$$\int N(y)\,dy = \int M(x)\,dx + c.$$

II. Exact equations : $M(x,y)\,dx + N(x,y)\,dy = 0$, where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. To solve,

$$\text{integrate} \quad \frac{\partial f}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y):$$

$$f(x,y) = \int M(x,y)\,dx + \phi(y) = \int N(x,y)\,dy + \psi(x) = c.$$

III. Linear equations :

$$\frac{dy}{dx} + p(x)\,y = q(x).$$

The solution is given by

$$y = \frac{1}{P(x)}\int P(x)q(x)\,dx + \frac{c}{P(x)}, \qquad \text{where} \qquad P(x) = e^{\int p(x)\,dx}.$$

IV. Homogeneous equations :

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{dy}{dx} = f\left(\frac{x}{y}\right).$$

Substitute $y = ux$, $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$ or $x = vy$, $\dfrac{dx}{dy} = y\dfrac{dx}{dy} + v$, integrate the resulting expression, then resubstitute for $u$ or $v$.

V. Equations in the rational form

$$\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f},$$

where $a,\ b,\ c,\ d,\ e,\ f$ are constants such that $ae \neq bd$. Substitute

$$x = X - h, \quad y = Y - k, \quad \text{where} \quad h = \frac{bf - ce}{ae - bd}, \quad k = \frac{cd - af}{ae - bd},$$

to obtain the homogeneous equation

$$\frac{dY}{dX} = \frac{aX + bY}{dX + eY}.$$

This equation may be solved as in item IV above.

## LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

I. Homogeneous equation $ay'' + by' + cy = 0$. Substitute $y = e^{rx}$ to find the characteristic equation, and write its roots $r_1$ and $r_2$ :

$$ar^2 + br + c = 0; \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solution, $y_c$, is given by the following table.

| | |
|---|---|
| Case 1. $b^2 - 4ac > 0$, real and distinct roots. | $y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ |
| Case 2. $b^2 - 4ac = 0$, two real equal roots. | $y_c = (c_1 + c_2 x)e^{rx}$ |
| Case 3. $b^2 - 4ac < 0$, two complex roots. | $y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$, where $\alpha = \frac{-b}{2a}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$ |

II. Undetermined Coefficients : $ay'' + by' + cy = F(x)$.

1. Solve the homogeneous equation $ay'' + by' + cy = 0$ for $y_c$.
2. Assume a particular solution $y_p$ for any term in $F(x)$ that is one of three special types, as follows.
   A. For a polynomial of degree $N$, $y_p$ is a polynomial of degree $N$.
   B. For an exponential $ke^{p(x)}$, $y_p = Ae^{p(x)}$.
   C. For $j \cos \omega x + k \sin \omega x$, $y_p = A \cos \omega x + B \sin \omega x$.
3. If any term in $y_p$ found above appears in $y_c$, multiply the corresponding trial term by the power of $x$ just high enough that the resulting product contains no term of $y_c$.
4. For any term in $F(x)$ that is a product of terms of the three types listed above, let $y_p$ be the product of the corresponding trial solutions.
5. Substitute $y_p$ into the given O.D.E. and evaluate the unknown coefficients by equating like terms.
6. The general solution of the given equation is

$$y = y_c + y_p.$$

Evaluate the two arbitrary constants $c_1$ and $c_2$ in the general solution $y$ by applying boundary/initial conditions, as appropriate.

III. Variation of Parameters : $ay'' + by' + cy = F(x)^*$.

1. Write the solution of the homogeneous equation $ay'' + by' + cy = 0$ in the form

$$y_c = c_1 y_1 + c_2 y_2.$$

2. Assume a particular solution $y_p$ for any term in the form

$$y_p = u_1 y_1 + u_2 y_2,$$

where $u_1$ and $u_2$ are functions determined by the conditions

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = G(x) \text{ where } G(x) = \frac{F(x)}{a}.$$

3. The solution of this system of equations is given by

$$u_1' = -\frac{y_2 G}{W} \quad \text{and} \quad u_2' = \frac{y_1 G}{W},$$

where $W$ denotes the Wronskian determinant:

$$W = W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

4. Integrate to find the unknown functions:

$$u_1(x) = -\int \frac{y_2(x)G(x)}{W(x)}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)G(x)}{W(x)}\,dx.$$

5. Form the particular solution $y_p = u_1 y_1 + u_2 y_2$.
6. The general solution of the given equation is

$$y = y_c + y_p.$$

Evaluate the two arbitrary constants $c_1$ and $c_2$ in the general solution $y$ by applying boundary/initial conditions, as appropriate.

IV. Laplace Transforms : $ay'' + by' + cy = F(x)$, $y(0) = y_0$ and $y'(0) = y_0'$.

1. Take the Laplace transform (see pages 9-10) of both sides of the given equation, using the given initial conditions. The result is an algebraic equation for $F(s)$, the Laplace transform of the solution.
2. Solve the algebraic equation obtained in Step 1 to find $F(s)$ explicitly.
3. The inverse Laplace transform of $F(s)$ is the solution of the given initial value problem.

*The coefficients may also be functions, i.e., $b = b(x)$ and $c = c(x)$.

## VECTOR ALGEBRA

1. Scalar and vector products. Given vectors

$$\mathbf{a} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 + \beta_3\mathbf{e}_3 = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$
$$\mathbf{c} = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 + \gamma_3\mathbf{e}_3 = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k},$$

let $\phi$ be the angle from **a** to **b**. The scalar (dot) product of **a** and **b** is the scalar

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\phi = a_xb_x + a_yb_y + a_zb_z.$$

The vector (cross) product of **a** and **b** is the vector of magnitude

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\phi,$$

perpendicular to **a** and **b** and in the direction of the axial motion of a right-handed screw turning **a** into **b.** In coordinate form,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_2\times\mathbf{e}_3 & \alpha_1 & \beta_1 \\ \mathbf{e}_3\times\mathbf{e}_1 & \alpha_2 & \beta_2 \\ \mathbf{e}_1\times\mathbf{e}_2 & \alpha_3 & \beta_3 \end{vmatrix}$$
$$= \begin{vmatrix} \mathbf{i} & a_x & b_x \\ \mathbf{j} & a_y & b_y \\ \mathbf{k} & a_z & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\mathbf{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\mathbf{k}$$
$$= (a_yb_z - a_zb_y)\mathbf{i} + (a_zb_x - a_xb_z)\mathbf{j} + (a_xb_y - a_yb_x)\mathbf{k}.$$

The box product of **a**, **b**, and **c** is the scalar quantity given by

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} \equiv [\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{bac}] = -[\mathbf{cba}] = -[\mathbf{acb}]$$
$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}.$$

The product of two box products is given by

$$[\mathbf{abc}][\mathbf{def}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix}.$$

A special case gives Gram′s determinant :

$$[\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$
$$= [(\mathbf{a}\times\mathbf{b})(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a})]$$
$$= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{c}) + 2(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{c})^2 - (\mathbf{b}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})^2 - (\mathbf{c}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})^2.$$

## VECTOR ALGEBRA (Continued)

The vector triple product of vectors **a, b,** and **c** is given by

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \begin{vmatrix} \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \end{vmatrix}.$$

Three additional formulas for scalar and vector products:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

$$(\mathbf{a}\times\mathbf{b})^2 = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2$$

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a} = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}.$$

## VECTOR ANALYSIS

Differential operators

$$\nabla\Phi(x,y,z) = \mathrm{grad}\,\Phi(x,y,z) \equiv \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$$

$$\nabla\cdot\mathbf{F}(x,y,z) = \mathrm{div}\,\mathbf{F}(x,y,z) \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\begin{aligned}\nabla\times\mathbf{F} &= \mathrm{curl}\,\mathbf{F}(x,y,z) \\ &\equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}\end{aligned}$$

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\mathbf{F} &= G_x\frac{\partial\mathbf{F}}{\partial x} + G_y\frac{\partial\mathbf{F}}{\partial y} + G_z\frac{\partial\mathbf{F}}{\partial z} \\ &= (\mathbf{G}\cdot\nabla F_x)\mathbf{i} + (\mathbf{G}\cdot\nabla F_y)\mathbf{j} + (\mathbf{G}\cdot\nabla F_z)\mathbf{k}\end{aligned}$$

$$\nabla^2 = (\nabla\cdot\nabla) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \quad \text{(Laplace operator)}$$

$$\begin{aligned}\text{div grad } \Phi &= \nabla\cdot(\nabla\Phi) = \nabla^2\Phi \\ \text{grad div } \mathbf{F} &= \nabla(\nabla\cdot\mathbf{F}) = \nabla^2\mathbf{F} + \nabla\times(\nabla\times\mathbf{F}) \\ \text{curl curl } \mathbf{F} &= \nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} \\ \text{curl grad } \Phi &= \nabla\times(\nabla\Phi) = 0 \\ \text{div curl } \mathbf{F} &= \nabla\cdot(\nabla\times\mathbf{F}) = 0 \\ \nabla(\Phi\Psi) &= \Psi\nabla\Phi + \Phi\nabla\Psi \\ \nabla^2(\Phi\Psi) &= \Psi\nabla^2\Phi + 2(\nabla\Phi)\cdot(\nabla\Psi) + \Phi\nabla^2\Psi\end{aligned}$$

## VECTOR ANALYSIS (Continued)

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$$
$$\nabla\cdot(\Phi\mathbf{F}) = \Phi\nabla\cdot\mathbf{F} + (\nabla\Phi)\cdot\mathbf{F}$$
$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\nabla\times\mathbf{F} - \mathbf{F}\cdot\nabla\times\mathbf{G}$$
$$(\mathbf{G}\cdot\nabla)(\Phi\mathbf{F}) = \mathbf{F}(\mathbf{G}\cdot\nabla\Phi) + \Phi(\mathbf{G}\cdot\nabla)\mathbf{F}$$
$$\nabla\times(\Phi\mathbf{F}) = \Phi\nabla\times\mathbf{F} + (\nabla\Phi)\times\mathbf{F}$$
$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F})$$

Cylindrical: $x = r\cos\theta, y = r\sin\theta, z = z$

$$\text{grad } \Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{k}}$$
$$\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$
$$\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \frac{1}{r}\hat{\mathbf{r}} & \hat{\theta} & \frac{1}{r}\hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$
$$\text{Laplacian } \Phi = \nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Spherical: $x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$

$$\text{grad } \Phi = \nabla\Phi = \frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{1}{\rho\sin\phi}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$$
$$\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2F_\rho)}{\partial\rho} + \frac{1}{\rho\sin\phi}\frac{\partial(\sin\phi\, F_\phi)}{\partial\phi} + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial\theta}$$
$$\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \frac{1}{\rho^2\sin\phi}\hat{\rho} & \frac{1}{\rho\sin\phi}\hat{\phi} & \frac{1}{\rho}\hat{\theta} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ F_\rho & \rho F_\phi & \rho\sin\phi\, F_\theta \end{vmatrix}$$
$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2\Phi}{\partial\theta^2}$$

Integral Theorems

$\int_V \nabla\cdot\mathbf{F}(\mathbf{r})\,dV = \int_S \mathbf{F}(\mathbf{r})\cdot d\mathbf{A}$ (Divergence theorem)

$\int_V \nabla\Phi\cdot\nabla\Psi\,dV + \int_V \Psi\nabla^2\Phi\,dV = \int_S(\Psi\nabla\Phi)\cdot d\mathbf{A}$ (Green's theorem)

$\int_V(\Psi\nabla^2\Phi - \Phi\nabla^2\Psi)\,dV = \int_S(\Psi\nabla\Phi - \Phi\nabla\Psi)\cdot d\mathbf{A}$ (symmetric form)

$\int_V \nabla^2\Phi\,dV = \int_S \nabla\Phi\cdot d\mathbf{A}$ (Gauss' theorem)

$\int_S[\nabla\times\mathbf{F}(\mathbf{r})]\cdot d\mathbf{A} = \oint_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ (Stokes' theorem)

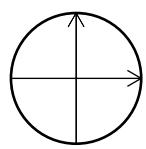
# FUNDAMENTALS OF SIGNALS AND NOISE

Complex numbers ($j^2 = -1$)

$$\begin{aligned} a + bj &= Re^{j\theta}, \qquad \text{where} \\ R &= \sqrt{a^2 + b^2}, \\ \theta &= \begin{cases} \arctan\left(\frac{b}{a}\right), & a > 0; \\ \arctan\left(\frac{b}{a}\right) \pm \pi, & a < 0; \\ \frac{\pi}{2}, & a = 0, b > 0; \\ -\frac{\pi}{2}, & a = 0, b < 0. \end{cases} \end{aligned}$$

$$\begin{aligned} Ae^{j\theta} \cdot Be^{j\phi} &= ABe^{j(\theta+\phi)} \\ e^{j\theta} &= e^{j(\theta + 2n\pi)}, \quad n = \pm 1, \pm 2, \ldots \end{aligned}$$

Unit phasor




Euler Identity

$$e^{\pm jx} = \cos x \pm j \sin x; \qquad \begin{cases} \cos x &= \frac{e^{jx} + e^{-jx}}{2}; \\ \sin x &= \frac{e^{jx} - e^{-jx}}{2j}. \end{cases}$$

Fourier series (Periodic signals. For the real number form, see p. 13.)

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_1 t}, \qquad \text{where} \\ c_n &= \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} x(t) e^{-j2\pi n f_1 t}\, dt, \\ T_1 &= \text{ period}, \\ f_1 &= \frac{1}{T_1} = \text{fundamental frequency}, \end{aligned}$$

(For real signals, $c_{-n} = c_n^*$ (complex conjugate).)

Fourier transform (aperiodic signals)

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt; \qquad x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi tf}\,df.$$

The correspondence between $x(t)$ and $X(f)$ is denoted by $x(t) \leftrightarrow X(f)$.

Transform of Fourier Series

$$\text{If } x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_1 t}, \quad \text{then } X(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_1).$$

Convolution ($*$) and Correlation ($\star$)

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau \equiv \int_{-\infty}^{\infty} y(\tau)x(t-\tau)\,d\tau = y * x$$

$$x \star y = \int_{-\infty}^{\infty} x(\tau)y(t+\tau)\,d\tau$$

Properties of the delta function

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)x(t)\,dt = x(t_0)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Parseval theorem (Average Power in a periodic signal)

$$\overline{P} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Rayleigh theorem (Total Energy in an aperiodic signal)

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

Fourier theorems

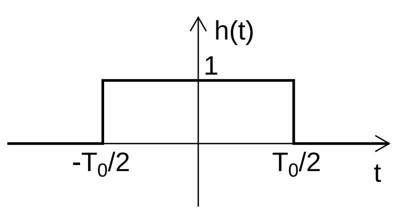
If $x(t) \leftrightarrow X(f)$ is a *transform pair*, then

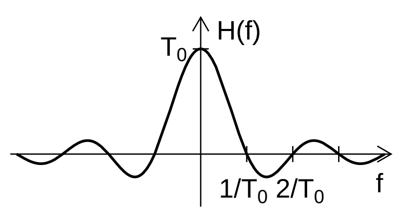
| | |
|---|---|
| LINEARITY | $ax(t) + by(t) \leftrightarrow aX(f) + bY(f)$ |
| SHIFT/MODULATION | $x(t - t_0) \leftrightarrow X(f)e^{-j2\pi t_0 f}$ |
| | $x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$ |
| PRODUCT/CONVOLUTION | $x(t)y(t) \leftrightarrow X * Y$ |
| | $x * y \leftrightarrow X(f)Y(f)$ |
| SCALE | $x(at) \leftrightarrow \frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$ |
| DIFFERENTIATION | $\frac{d}{dt}x(t) \leftrightarrow j2\pi f X(f)$ |
| INTEGRATION | $\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{X(f)}{j2\pi f}.$ |

Fourier pairs

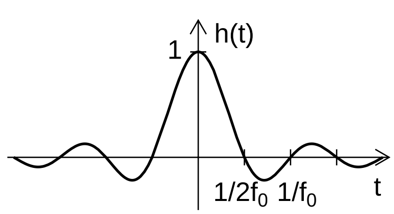
Time Domain ↔ Frequency Domain

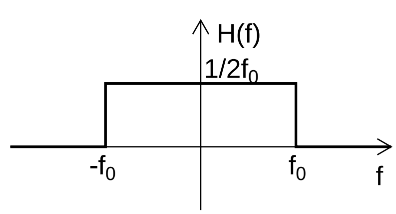
$$h(t) = 1 - u_{\frac{T_0}{2}}(|t|) \quad \leftrightarrow \quad H(f) = \frac{\sin(\pi T_0 f)}{\pi f}$$



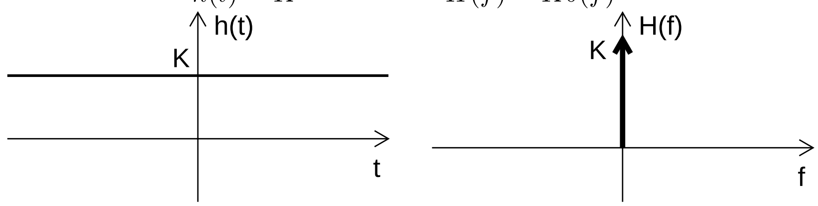

$$h(t) = \frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \quad \leftrightarrow \quad H(f) = \frac{1}{2f_0}(1 - u_{f_0}(|f|))$$



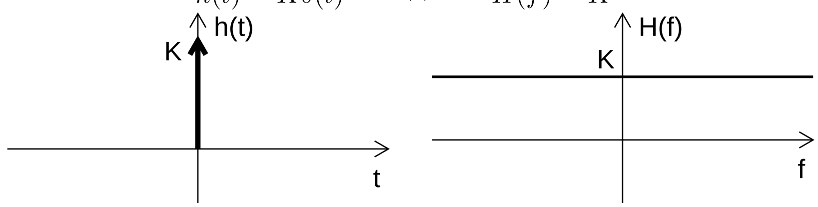



Fourier pairs Continued

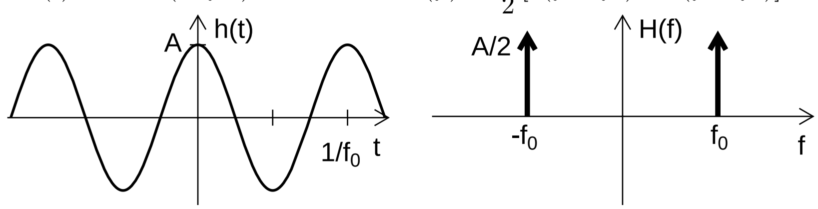
| Time Domain | | Frequency Domain |
|---|---|---|
| $h(t) = K$ | $\leftrightarrow$ | $H(f) = K\delta(f)$ |



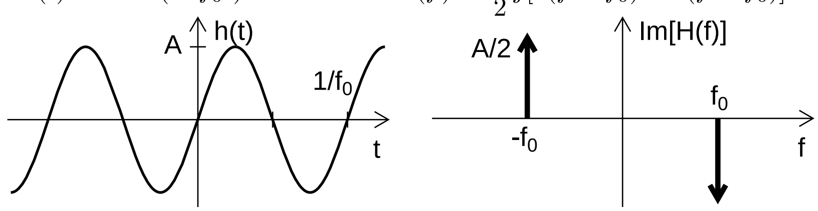

$$h(t) = K\delta(t) \quad \leftrightarrow \quad H(f) = K$$



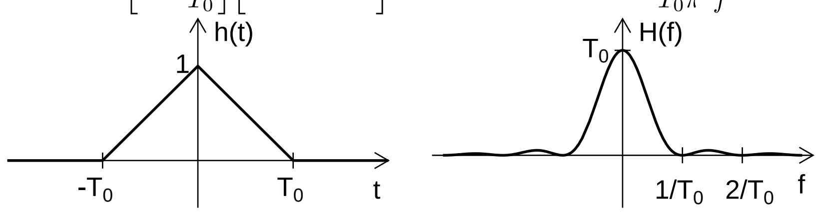

$$h(t) = A\cos(2\pi f_0 t) \quad \leftrightarrow \quad H(f) = \frac{A}{2}[\delta(f+f_0) + \delta(f-f_0)]$$




$$h(t) = A\sin(2\pi f_0 t) \quad \leftrightarrow \quad H(f) = \frac{A}{2}j[\delta(f+f_0) - \delta(f-f_0)]$$

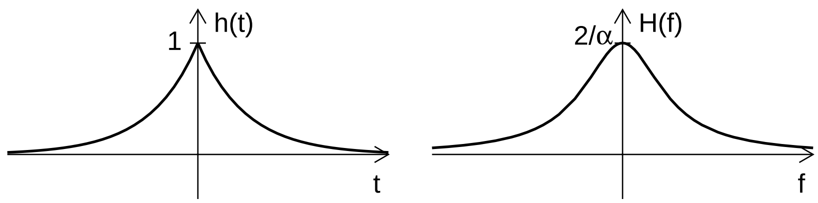



$$h(t) = \left[1 - \frac{|t|}{T_0}\right]\left[1 - u_{T_0}(|t|)\right] \quad \leftrightarrow \quad H(f) = \frac{\sin^2(\pi T_0 f)}{T_0 \pi^2 f^2}$$

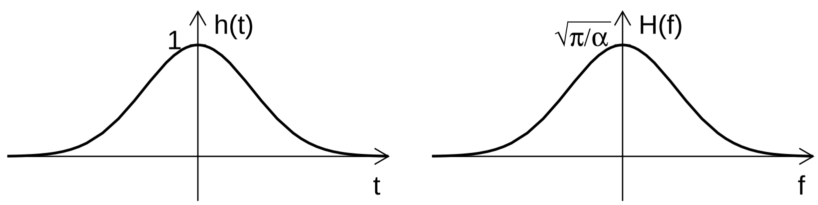


Fourier pairs Continued

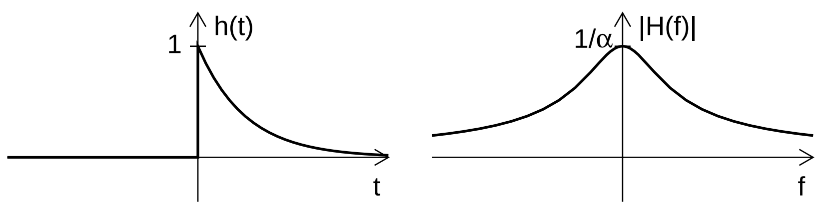
| Time Domain | | Frequency Domain |
|---|---|---|
| $h(t) = e^{-\alpha\|t\|}$ | $\leftrightarrow$ | $H(f) = \dfrac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$ |



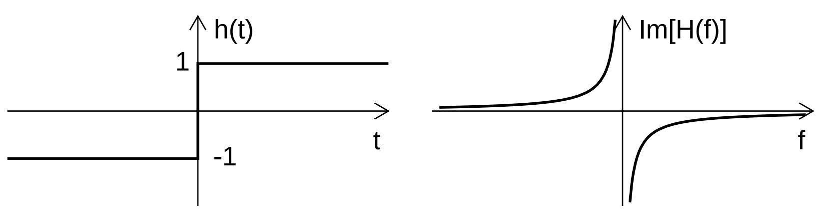

$$h(t) = \exp(-\alpha t^2) \quad \leftrightarrow \quad H(f) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{-\pi^2 f^2}{\alpha}\right)$$




$$h(t) = \begin{cases} e^{-\alpha t}, & t > 0 \\ 0, & t \leq 0 \end{cases} \quad \leftrightarrow \quad H(f) = \frac{1}{\alpha + 2\pi j f}$$



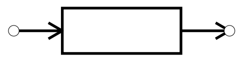

$$h(t) = \operatorname{sgn}(t) \quad \leftrightarrow \quad H(f) = \frac{-j}{\pi f}$$

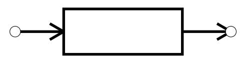


Linear time − invariant systems

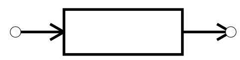
Basic Properties

If

$x(t)$  $y(t)$

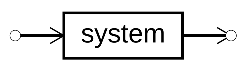
$g(t)$  $q(t)$

then

$ax(t-t_0)+bg(t-t_0)$  $ay(t-t_0)+bq(t-t_0)$

Transform properties

| | Input | Output |
|---|---|---|
| Time Domain | $\delta(t)$ | $h(t)$ |
| | $x(t)$ | $y(t) = x(t) * h(t)$ |




| | | |
|---|---|---|
| Frequency Domain | $X(f)$ | $Y(f) = X(f) \cdot H(f)$ |

$h(t) \leftrightarrow H(f)$ $\quad h(t) \equiv$ impulse response; $\quad H(f) \equiv$ transfer function

Random signals

$$\overline{x} = \int_{-\infty}^{\infty} x p(x)\, dx \qquad \text{Mean value}$$

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 p(x)\, dx \qquad \text{Second moment}$$

$$< x, x >= \|x\|^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt \qquad \text{Norm square}$$

$$< x > = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt \qquad \text{Time average}$$

For an ergodic process:
Second moment = Norm square and Mean value = Time average.

Autocorrelation

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)\,dt$$

Power Spectral Density

The power spectral density $G_{xx}$ is the Fourier transform of the autocorrelation function:

$$R_{xx}(t) = \int_{-\infty}^{\infty} G_{xx}(f)e^{j2\pi tf}\,df \quad \leftrightarrow \quad G_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(t)e^{-j2\pi ft}\,dt$$

Noise power ($G_{xx}$ denotes the power spectral density)

$$\overline{P}_{\text{Total}} = \int G_{xx}\,df = R_{xx}(0) = \overline{x^2} = (\overline{x})^2$$

$$\overline{P}_{ac} = N = \sigma_x^2 = \text{ noise power}$$

$$\overline{P}_{dc} = R_{xx}(\infty) = (\overline{x})^2$$

Noise bandwidth ($G_{xx}$ denotes the power spectral density)

$$G_{yy} = G_{xx} \cdot |H|^2 \qquad \text{(linear system)}$$

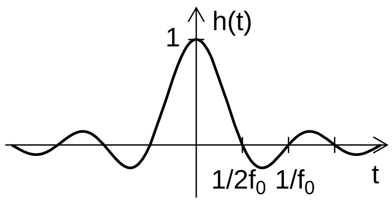
$$N_y = \int_{-\infty}^{\infty} G_{xx} \cdot |H|^2\,df = \text{ noise power output}$$

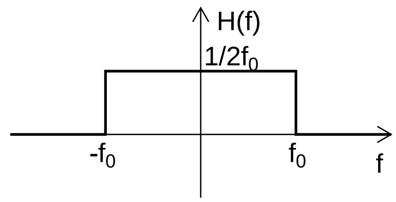
$$N = n_0 B_N = \text{ noise power, where } \frac{n_0}{2} = \text{ white noise}$$

$$B_N = \int_{-\infty}^{\infty} |H|^2\,df = \text{ noise bandwidth}$$

Ideal filter (low − pass)

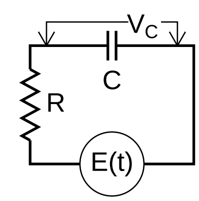
$$h(t) = \frac{\sin(2\pi f_0 t)}{2\pi f_0} \qquad \leftrightarrow \qquad H(f) = \frac{1 - u_{f_0}(|f|)}{2 f_0}$$

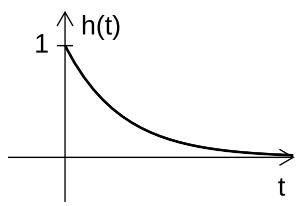


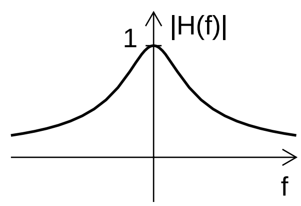




RC filter (low − pass)

$$V_C = h(t){*}E(t) \qquad h(t) = \frac{1}{RC} e^{-t/RC} \qquad \leftrightarrow \qquad H(f) = \frac{1}{1 + 2\pi j f RC}$$

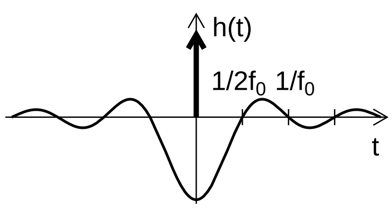


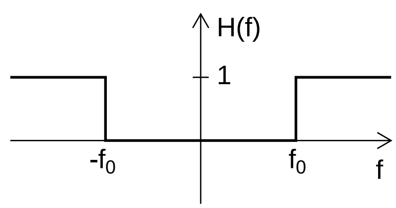




Ideal filter (high − pass)

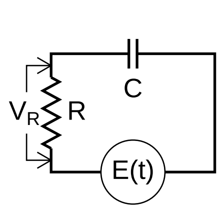
$$h(t) = \delta(t) - \frac{\sin(2\pi f_0 t)}{\pi t} \qquad \leftrightarrow \qquad H(f) = u_{f_0}(|f|)$$

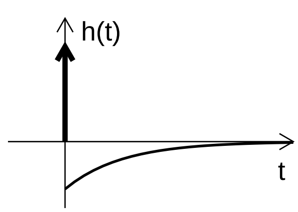


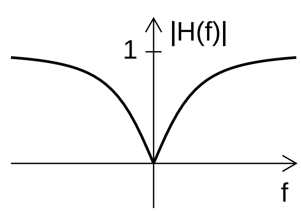




RC filter (high − pass)

$$V_R = h(t){*}E(t) \qquad h(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} \qquad \leftrightarrow \qquad H(f) = \frac{2\pi j f RC}{1 + 2\pi j f RC}$$

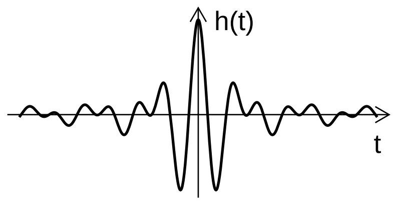


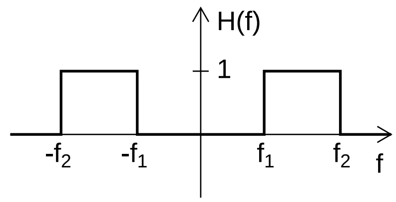



Ideal filter (band − pass)

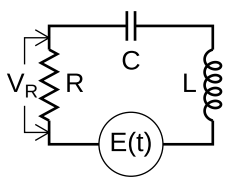
$$h(t) = \frac{\sin(2\pi f_2 t)}{\pi t} - \frac{\sin(2\pi f_1 t)}{\pi t} \qquad \leftrightarrow \qquad H(f) = u_{f_2}(|f|) - u_{f_1}(|f|)$$

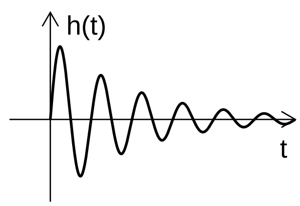


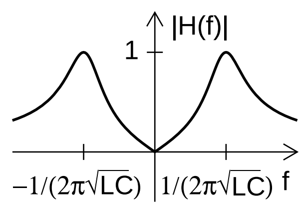



LRC filter (band − pass)

$$V_R = h(t) * E(t) \qquad h(t) \qquad \leftrightarrow \qquad H(f) = \frac{2\pi j f RC}{1 - (2\pi f)^2 LC + 2\pi j f RC}$$








Ideal sampling

Sampled signal

$$x_\delta(t) = x(t) \cdot s_\delta(t) \qquad \text{where} \qquad s_\delta = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Spectrum after sampling

$$X_\delta(f) = X(f) * S_\delta(f) \qquad \text{where} \qquad S_\delta(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

Sampling theorem

If

$$T_s < \frac{1}{2W} \qquad \text{or} \qquad f_s > 2W \qquad (\text{“Nyquist rate”})$$

then the signal can be reconstructed with a filter of bandwidth $B$, given by:

$$W < B < f_s - W.$$