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RESONANCES IN NONSTATIONARY, NONLINEAR, MULTIDEGREE-OF-FREEDOM SYSTEMS

by
BRIJ N. AGRAWAL
COMSAT Laboratories
Clarksburg, Maryland
and
R. M. EVAN-IWANOWSKI
Syracuse University
Syracuse, New York

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RESONANCES IN NONSTATIONARY, NONLINEAR, MULTIDEGREE-OF-FREEDOM SYSTEMS*

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Brij N. Agrawal**
COMSAT Laboratories
Clarksburg, Maryland

and

R. M. Evan-Iwanowski†
Syracuse University
Syracuse, New York

ABSTRACT

The asymptotic method for determining resonant responses of nonstationary nonlinear systems is presented. Resonance conditions, resonance coefficients, and higher order resonances are discussed. The first asymptotic approximation nonstationary solution is obtained for general resonances. A gyroscopic system is analyzed for combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$. Using the general solution, nonstationary and stationary responses and stability conditions are obtained. The numerical results indicate that the change in the rate of variation of the frequency of excitation may shift the nonstationary response from one stable mode to another stable mode.

NOMENCLATURE

A_j^m = nonoscillatory function

a_j = amplitude of the jth mode

B_j^m = nonoscillatory function

b = linear stiffness of bearing C in the α and β directions (Figure 1)

$\bar{b}_2$ = quadratic nonlinear coefficient of bearing C in the β direction

$\bar{b}_3$ = cubic nonlinear coefficient of bearing C in the β direction

D = $\nu(\tau)[\partial/\partial\theta] + \Sigma_{\ell=1}^n \omega_\ell[\partial/\partial\psi_\ell]$

$\bar{e}$ = eccentricity of the rotor, D, with respect to the rotation axis (Figure 1)

F_{cjk}^m = coefficient of cos $(k_0\theta + \Sigma_{r=1}^n k_r\psi_r)$ in $f_j^m = F_{cjk_0\ldots k_n}^m$

F_{sjk}^m = coefficient of sin $(k_0\theta + \Sigma_{r=1}^n k_r\psi_r)$ in $f_j^m = F_{sjk_0\ldots k_n}^m$

F_{cjj}^1 = coefficient of cos ψ_j in f_j^1

F_{sjj}^1 = coefficient of sin ψ_j in f_j^1

f_j = perturbation force in the jth mode

I = moment of inertia of the rotor with respect to the transverse axis passing through the rotor's CG

I_p = moment of inertia of the rotor with respect to the axis of symmetry

I_1 = moment of inertia of the rotor with respect to an axis passing through the lower bearing perpendicular to the symmetry axis, $I_1 = I + ML_2^2$

L = upper limit of time interval

L_1 = distance from bearing O to bearing C (Figure 1)

L_2 = distance from bearing O to the rotor's CG

M = mass of the rotor

m = order of the asymptotic approximation

n = number of degrees of freedom

P = weight of the rotor

U_j^m = periodic function of angles θ and $\psi_1, \ldots, \psi_n$

X_j = normalized coordinate

X_{j0} = first asymptotic approximation of X_j, $X_{j0} = a_j \cos \psi_j$

$\dot{X}_{j0}$ = first asymptotic approximation of $\dot{X}_j$, $\dot{X}_{j0} = -a_j\omega_j \sin \psi_j$

α, β = angles defining the position of the axis of gyroscope in the fixed coordinate axis OXYZ (Figure 1)

ε = small positive parameter

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**Member of the technical staff.

†Professor, Dept. of Mechanical and Aerospace Engineering.

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θ = phase angle of the external periodic excitation; angle of rotation of the rotor in Figure 1
ν = instantaneous frequency of the external excitation $\nu = \dot{\theta}$
τ = slow time, $\tau = \varepsilon t$, which varies from 0 to L
τ^* = specific time in the range of τ, $\tau^* \varepsilon [0,L]$
ψ_j = phase angle of the jth mode
Ω = angle between the axis of symmetry of the rotor system and the rotation axis
ω_j = natural frequency of the jth mode of the linear system
$\cdot$ = d/dt, or differentiation with respect to time t

Subscripts

c = coefficient of the cosine function
j = jth mode
k_r = coefficient of ψ_r in the harmonic function associated with the term
k_0 = coefficient of θ in the harmonic function associated with the term
s = coefficient of the sine function

Superscript

1,2,...,m = order of the asymptotic approximation for $A_j, B_j, f_j, N_j, U_j, k_r$; power for the remaining symbols except g's

INTRODUCTION

Nonstationary mechanical systems are those systems whose parameters, such as mass, stiffness, natural frequency, and external perturbation frequency, are time dependent. These systems are frequently encountered in practical applications such as transition resonance of turbo engines, vibration testing of space vehicles, and variable mass of a rocket during launch.

Lewis[1] was the first to present a solution for the response of a nonstationary, linear, single-degree-of-freedom mechanical system subjected to an excitation whose frequency is a linear function of time. An outstanding contribution in this field of mechanics was also made by the Russian school. In particular, Mitropolskii extended the asymptotic method to nonstationary problems, although he did not mention combination resonances in his monograph.[2] Combination resonances and related concepts, such as resonance coefficients and resonance conditions in stationary nonlinear systems, are discussed by Mettler,[3] who applied the averaging method, and by Leiss,[4] who used the asymptotic method. An exhaustive bibliography on the subject of nonstationary systems can be found in a survey paper by Evan-Iwanowski.[5]

In this paper, the asymptotic method is presented to determine the resonant response of nonstationary, nonlinear, multidegree-of-freedom systems for general resonances such as combination resonances. The first asymptotic approximation solution is obtained for the general resonance. The concept of virtual work is applied to define resonance, resonance coefficients, and higher order resonances. A gyroscopic system exhibiting combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ is analyzed. The general solution is used to obtain the nonstationary response, stationary response, and stability conditions for these resonances. Nonstationary responses are obtained for the various functions of the frequency of excitation. The details of the work presented in this paper are given in Reference 6.

ASYMPTOTIC METHOD

The equations of motion of an n-degree-of-freedom, asymptotic, holonomic mechanical system can be normalized and written in the following form:

$$\ddot{X}_j + \omega_j^2(\tau)\, X_j = \varepsilon f_j(\tau,\theta,X_1,\ldots,X_n,\dot{X}_1,\ldots,\dot{X}_n) \qquad j = 1,\ldots,n \tag{1}$$

In equation (1), the terms which are functions of τ are varying slowly with time. The method presented in this paper requires that the system parameters vary slowly compared to a natural time unit, which is a time unit of the order of the vibration period. The time τ varies from 0 to L. Setting $\varepsilon = 0$ in equation (1) and assuming that τ is a parameter results in an equation, called an unperturbed equation, which can be solved as follows:

$$X_j = a_j \cos \psi_j \qquad \dot{a}_j = 0 \qquad \dot{\psi}_j = \omega_j \qquad j = 1,\ldots,n \tag{2}$$

When $\varepsilon \neq 0$, i.e., in the presence of perturbation, higher harmonics may appear in the solutions and the natural frequency may depend on the amplitude. Furthermore, various resonances may take place, and the variation of $\omega_j(\tau)$ and $\nu(\tau)$ with slow time, τ, will result in additional phenomena which are not observed in nonlinear stationary systems. Taking into account these physical arguments and keeping in mind that when $\varepsilon \to 0$ the solution should be represented by equation (2), we use the following form to solve equation (1) for the mth approximation:

$$X_j = a_j(\tau)\cos\psi_j(\tau) + \sum_{i=1}^{m} \varepsilon^i \times U_j^i(\tau,a_1,\ldots,a_n,\theta,\psi_1,\ldots,\psi_n) \tag{3}$$

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where U_j^i are unknown functions, periodic in θ and $\psi_1, \ldots, \psi_n$ and dependent on $a_1, \ldots, a_n$. The functions a_j and ψ_j are determined from the following equations:

$$\dot{a}_j = \sum_{i=1}^{m} \varepsilon^i \times A_j^i(\tau, a_1, \ldots, a_n, \theta, \psi_1, \ldots, \psi_n) \qquad (4a)$$

$$\dot{\psi}_j = \omega_j + \sum_{i=1}^{m} \varepsilon^i \times B_j^i(\tau, a_1, \ldots, a_n, \theta, \psi_1, \ldots, \psi_n) \qquad (4b)$$

where A_j^i and B_j^i are nonoscillatory functions.

U_j^i, A_j^i, and B_j^i are selected so that, after a_j and ψ_j are replaced with the functions defined in equation (4), equation (3) will satisfy equation (1) up to ε^m. The coefficients A_j^i and B_j^i are also unknowns in the determination of X_j. Obviously, equation (1) is insufficient to determine the unique values of these coefficients. To obtain unique values, an additional condition is necessary; i.e., U_j^i must be finite.

The asymptotic method presented here is similar to the asymptotic method developed by Mitropolskii for nonstationary systems. The essential difference lies in the form in which the solution is sought. In the present method, this form is the same for all resonances; in Mitropolskii's method, it changes for different resonances. The former approach, as will be clear later, is a unified approach for all resonances, and makes it possible to obtain resonance coefficients and conditions.

After determining the first and second derivatives of X_j with respect to time t by using equations (3) and (4) and substituting $\dot{X}_j$ and $\ddot{X}_j$ in the left-hand side of equation (1), we obtain

$$\begin{aligned}\ddot{X}_j + \omega_j^2 X_j = \varepsilon\Big[&\cos\psi_j\left(DA_j^1 - 2a_j B_j^1 \omega_j\right) \\ &- \sin\psi_j\left(a_j \frac{\partial \omega_j}{\partial \tau} + 2A_j^1\omega_j + a_j DB_j^1\right) + D^2U_j^1 + \omega_j^2 U_j^1\Big] \\ &+ \sum_{i=2}^{m} \varepsilon^i \Big[\cos\psi_j\left(DA_j^i - 2a_j B_j^i \omega_j\right) - \sin\psi_j\left(2A_j^i\omega_j + a_j DB_j^i\right) + D^2U_j^i + N_j^i + \omega_j^2 U_j^i\Big]\end{aligned} \qquad (5)$$

where the differential operators D, N_j^m, and $U_{j\ell}$ are defined as follows:

$$D = \nu(\tau)\frac{\partial}{\partial\theta} + \sum_{\ell=1}^{n} \omega_\ell \frac{\partial}{\partial\psi_\ell} \qquad (6a)$$

$$\begin{aligned}N_j^i = \cos\psi_j\Bigg[&\frac{\partial A_j^{i-1}}{\partial\tau} - a_j \sum_{\ell=1}^{i-1} B_j^\ell B_j^{i-\ell} + \sum_{\ell=1}^{i-1}\sum_{k=1}^{n}\left(\frac{\partial A_j^\ell}{\partial a_k} A_k^{i-\ell} + \frac{\partial A_j^\ell}{\partial \psi_k} B_j^{i-\ell}\right)\Bigg] \\ - \sin\psi_j\Bigg[&a_j \frac{\partial B_j^{i-1}}{\partial\tau} + 2\sum_{\ell=1}^{i-1} A_j^\ell B_j^{i-\ell} + a_j \sum_{\ell=1}^{i-1}\sum_{k=1}^{n}\left(\frac{\partial B_j^\ell}{\partial a_k} A_k^{i-\ell} + \frac{\partial B_j^\ell}{\partial\psi_k} B_k^{i-\ell}\right)\Bigg] + \frac{\partial DU_j^{i-1}}{\partial\tau} + \frac{\partial U_{j,i-1}}{\partial\tau} \\ &+ DU_{ji} + \sum_{\ell=1}^{i-1}\sum_{k=1}^{n}\left(\frac{\partial DU_j^\ell}{\partial a_k} A_k^{i-\ell} + \frac{\partial DU_j^\ell}{\partial\psi_k} B_k^{i-\ell}\right) + \sum_{\ell=1}^{i-1}\sum_{k=1}^{n}\left(\frac{\partial U_{j\ell}}{\partial a_k} A_k^{i-\ell} + \frac{\partial U_{j\ell}}{\partial\psi_k} B_k^{i-\ell}\right)\end{aligned} \qquad (6b)$$

$$U_{j\ell} = \frac{\partial U_j^{\ell-1}}{\partial\tau} + \sum_{p=1}^{\ell-1}\sum_{k=1}^{n}\left(\frac{\partial U_j^p}{\partial a_k} A_k^{\ell-p} + \frac{\partial U_j^p}{\partial\psi_k} B_k^{\ell-p}\right) \qquad (7)$$

Expanding f_j in the right-hand side of equation (1) into Taylor's series results in

$$\begin{aligned}&\varepsilon f_j(\tau, \theta, X_1, \ldots, X_n, \dot{X}_1, \ldots, \dot{X}_n) \\ &= \varepsilon\Bigg\{f_j(\tau, \theta, X_{10}, \ldots, X_{n0}, \ldots, \dot{X}_{10}, \ldots, \dot{X}_{n0}) + \sum_{k=1}^{\infty}\frac{1}{k!}\left[\left.\frac{\partial^k f_j}{\partial X_i^k}\right|_{X_i = X_{i0}} (\Delta X_i)^k + \left.\frac{\partial^k f_j}{\partial \dot{X}_i^k}\right|_{\dot{X}_i = \dot{X}_{i0}} (\Delta\dot{X}_i)^k\right]\Bigg\} \\ &= \sum_{i=1}^{\infty} \varepsilon^i f_j^i(\tau, \theta, a_1, \ldots, a_n, \psi_1, \ldots, \psi_n)\end{aligned} \qquad (8)$$

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where

$$X_{i_0} = a_i \cos \psi_i$$

$$\dot{X}_{i_0} = -a_i\omega_i \sin \psi_i$$

$$\Delta X_i = X_i - a_i \cos \psi_i$$

$$= \varepsilon U_i^1 + \varepsilon^2 U_i^2 + \varepsilon^3 U_i^3 + \ldots$$

$$\Delta \dot{X}_i = \dot{X}_i + a_i\omega_i \sin \psi_i$$

$$= \varepsilon[\quad] + \varepsilon^2[\quad] + \ldots$$

$$f_j^1 = f_j(\tau,\theta,X_{1_0},\ldots,X_{n_0},\dot{X}_{1_0},\ldots,\dot{X}_{n_0}) \quad (9)$$

Equating the coefficients of the same power of ε, up to and including mth-order terms, in equations (5) and (8), we obtain

$$D^2U_j^1 + \omega_j^2 U_j^1$$

$$= \sin \psi_j\left(a_j \frac{\partial \omega_j}{\partial \tau} + 2A_j^1\omega_j + a_j DB_j^1\right)$$

$$- \cos \psi_j (DA_j^1 - 2a_j B_j^1 \omega_j)$$

$$+ f_j(\tau,\theta,X_{1_0},\ldots,X_{n_0},\dot{X}_{1_0},\ldots,\dot{X}_{n_0}) \quad (10.1)$$

$$D^2U_j^2 + \omega_j^2 U_j^2$$

$$= \sin \psi_j (2A_j^2\omega_j + a_j DB_j^2)$$

$$- \cos \psi_j (DA_j^2 - 2a_j B_j^2 \omega_j)$$

$$+ f_j^2 - N_j^2 \quad (10.2)$$

$$\vdots$$

$$D^2U_j^m + \omega_j^2 U_j^m$$

$$= \sin \psi_j (2A_j^m\omega_j + a_j DB_j^m)$$

$$- \cos \psi_j (DA_j^m - 2a_j B_j^m \omega_j)$$

$$+ f_j^m - N_j^m \quad (10.m)$$

The steps leading to the mth-order approximation are as follows: 1. calculate U_j^1, A_j^1, and B_j^1 by solving equation (10.1) and constraining U_j^1 to exclude secular terms. 2. Substitute the values of U_j^1, A_j^1, and B_j^1 obtained in step 1 into $f_j^2 - N_j^2$ in equation (10.2); calculate U_j^2, A_j^2, and B_j^2 by solving equation (10.2) and constraining U_j^2 to exclude secular terms.

Proceeding in a similar manner, we determine the mth approximations as follows. The values of U_j^i, A_j^i, and B_j^i ($i = 1, 2, \ldots, m - 1$) obtained from the previous steps are substituted into $f_j^m - N_j^m$ in equation (10.m). U_j^m, A_j^m, and B_j^m are then calculated by solving equation (10.m) and constraining U_j^m to exclude secular terms. Substituting U_j^m, A_j^m, and B_j^m into equations (3) and (4) yields the mth asymptotic solution.

Resonances

Resonance is characterized by a large system response amplitude caused by a small perturbation force. This phenomenon can be explained in terms of virtual work; that is, it takes place when the virtual work done by the perturbing forces over a cycle of a particular mode is not equal to zero over a large time interval.

Consider the virtual work of the perturbing force εf_j along the virtual displacement corresponding to the mode of the first harmonic of X_j; i.e.,

$$\text{virtual work} = \varepsilon f_j \delta X_j$$

$$= \sum_{m=1}^{\infty} \varepsilon^m [f_j^m (\delta a_j \cos \psi_j - \delta\psi_j a_j \sin \psi_j)] \quad (11)$$

Expanding f_j^m into Fourier series results in

$$f_j^m = \sum_{k_0} \cdots \sum_{k_n} \Bigg[F^m_{cjk_0\ldots k_n} \times \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right) + F^m_{sjk_0\ldots k_n} \times \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right)\Bigg] \quad (12)$$

Henceforth, $F_{cjk_0\ldots k_n}$ and $F_{sjk_0\ldots k_n}$ will be referred to as F_{cjk} and F_{sjk}, respectively. Substituting f_j^m from equation (12) into equation (11) and averaging the virtual work over a large time interval, T, indicates that only nonperiodic terms will be nonzero. Hence, only those terms whose frequencies are equal to ω_j, i.e., whose indices k satisfy the following relationship

$$k_0\nu(\tau^*) + \sum_{r=1}^{n} k_r\omega_r(\tau^*) = \pm\, \omega_j(\tau^*) \quad (13)$$

for some time $\tau^* \varepsilon [0,L]$, will contribute to f_j^m.

The Fourier coefficients F_{sjk} and F_{cjk}, which correspond to the previous resonance relationship, are called resonance coefficients. Hence, in order to have resonance, two conditions should be satisfied. First, the resonance relationship $k_0\nu(\tau) + \Sigma_{r=1}^{n} k_r\omega_r(\tau^*) = \pm\, \omega_j(\tau^*)$ must be satisfied, and second, at least one of the corresponding resonance coefficients should be nonzero.

Clearly the resonance conditions may be satisfied by asymptotic approximations of various orders of ε. That is,

$$k_0^1\nu + \sum_{r=1}^{n} k_r^1\omega_r = \pm\, \omega_j$$

$$F_{cjk}^1 \neq 0 \quad \text{or} \quad F_{sjk}^1 \neq 0 \quad (14.1)$$

$$\vdots$$

$$k_0^\ell \nu + \sum_{r=1}^{n} k_r^\ell \omega_r = \pm\, \omega_j$$

$$F_{cjk}^\ell \neq 0 \quad \text{or } F_{sjk}^\ell \neq 0 \qquad (14.\ell)$$

where the superscripts indicate the order of ε of the asymptotic approximation.

Some of the resonances may be satisfied in more than one order of ε. Hence, the ℓth-order resonance may be defined as follows. Let us denote the elements of the sets of indices satisfying the ℓth-order resonance relationship as $\{k_r^\ell\}$. If $\{k_r^\ell\}$ is not contained in any $\{k_r^i\}_\ell$ where $i < \ell$, and if either the F_{cjk}^ℓ or F_{sjk}^ℓ or both are nonzero, then the conditions are satisfied for the existence of the ℓth-order resonance. This resonance relationship may be expressed as

$$k_0^\ell \nu + \sum_{r=1}^{n} k_r^\ell \omega_r = \pm\, \omega_j \qquad (15)$$

The resonance relationship expressed by equation (13) may be rewritten as

$$h_0 \nu(\tau^*) = \sum_{r=1}^{n} h_r \omega_r(\tau^*) \qquad (16)$$

where

$$h_0 = k_0$$
$$h_r = -k_r \pm \delta_{rj}$$

The resonance expressed by equation (16) may be considered to be a general resonance. Other types of resonance, which are special cases of equation (16), are listed in Table 1.

First Asymptotic Approximation Solution

By expanding $f_j(\tau,\ \theta,\ X_{10},\ .\ .\ .,\ X_{n0},\ \dot{X}_{10},\ .\ .\ .,\ \dot{X}_{n0})$ into a Fourier series and substituting the resulting values into equation (10.1), we obtain

$$\begin{aligned} D^2U_j^1 + \omega_j^2 U_j^1 = {} & \sin \psi_j\left(a_j \frac{\partial \omega_j}{\partial \tau} + 2A_j^1\omega_j + a_j DB_j^1\right) \\ & - \cos \psi_j\left(DA_j^1 - 2a_j B_j^1 \omega_j\right) \\ & + \sum_{k_0} \cdots \sum_{k_n}\left[F_{sjk}^1 \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right) + F_{cjk}^1 \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r\right)\right] \end{aligned} \qquad (17)$$

For U_j^1 to be finite, the right-hand side of equation (17) must not contain harmonics of frequency ω_j. Thus the terms containing harmonics of ω_j or secular terms should be set equal to zero. It should be noted that, in f_j^1, the harmonic terms whose frequency is ω_j for $\tau^*\varepsilon[0,L]$ contribute virtual work in equation (11);

Table 1. Resonance Types

Resonance Relationship	Type of Resonance
$h_0\nu = \sum_{r=1}^{n} h_r\omega_r$, $h_r \geq 0$	Combination Additive Resonance
$h_0\nu = \sum_{r=1}^{n} h_r\omega_r$, some $h_r \leq 0$	Combination Differential Resonance
$\sum_{r=1}^{n} h_r\omega_r = 0,\ h_0 = 0$	Internal Resonance
$\nu = \omega_j,\ h_0 = 1,\ h_r = \delta_{rj}$	Principal Resonance
$\frac{\nu}{h_j} = \omega_j$, $h_0 = 1,\ h_r = 0,\ r \neq j$	Subharmonic Resonance
$h_j = 2$	Parametric Resonance
$h_0\nu = \omega_j,\ h_r = \delta_{rj}$	Superharmonic Resonance
$\frac{h_0\nu}{h_j} = \omega_j$	Rational Resonances

hence these terms cause resonance. These same terms are secular terms in equation (17). Nonresonant and resonant cases will be discussed in the following paragraphs.

Nonresonance. If both resonance conditions are not satisfied, then the system is called nonresonant. This indicates that f_j^1 does not contain harmonic terms whose frequency is ω_j for $\tau^* \epsilon [0,L]$. Equating to zero the coefficients containing harmonics of ω_j in equation (17) results in

$$a_j \frac{\partial \omega_j}{\partial \tau} + 2A_j^1 \omega_j + a_j D B_j^1 + F_{sjj}^1 = 0$$
$$DA_j^1 - 2a_j B_j^1 \omega_j - F_{cjj}^1 = 0 \quad (18)$$

where F_{cjj}^1 and F_{sjj}^1 are the coefficients of $\cos \psi_j$ and $\sin \psi_j$, respectively. Solving equation (18) for A_j^1 and B_j^1 and substituting these values into equation (4) yields

$$\dot{a}_j = \epsilon\left[-\frac{1}{2}\frac{F_{sjj}^1}{\omega_j} - \frac{1}{2} a_j \frac{(\partial\omega_j/\partial\tau)}{\omega_j}\right]$$
$$\dot{\psi}_j = \omega_j - \epsilon \frac{F_{cjj}^1}{2a_j\omega_j} \quad (19)$$

Resonance. A system is resonant if both resonance conditions are satisfied. This indicates that f_j^1 contains harmonic terms whose frequency is ω_j for $\tau^* \epsilon [0,L]$. Hence,

$$k_0 \nu(\tau^*) + \sum_{r=1}^{n} k_r \omega_r(\tau^*) = \pm\, \omega_j(\tau^*) \quad (20)$$

Equating to zero coefficients containing harmonics whose frequency is ω_j for τ^* results in

$$a_j \frac{\partial \omega_j}{\partial \tau} + 2A_j^1 \omega_j + a_j D B_j^1 + F_{sjj}^1 \pm F_{sjk}^1 \times \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) \mp F_{cjk}^1 \times \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) = 0$$
$$DA_j^1 - 2a_j B_j^1 \omega_j - F_{cjj}^1 - F_{sjk}^1 \times \sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) - F_{cjk}^1 \times \cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right) = 0 \quad (21)$$

Solving equation (21) for A_j^1 and B_j^1 and substituting these values into equation (4) yields

$$\dot{a}_j = \epsilon\left[-\frac{1}{2}\frac{F_{sjj}^1}{\omega_j} - \frac{1}{2} a_j \frac{(\partial\omega_j/\partial\tau)}{\omega_j} - F_{sjk}^1 \frac{\cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)}{k_0\nu + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j} + F_{cjk}^1 \frac{\sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \pm \psi_j\right)}{k_0\nu + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j}\right]$$

$$\dot{\psi}_j = \omega_j + \epsilon\left[-\frac{1}{2}\frac{F_{cjj}^1}{a_j\omega_j} \mp F_{sjk}^1 \frac{\sin\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)}{a_j\left(k_0\nu + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)} \mp F_{cjk}^1 \frac{\cos\left(k_0\theta + \sum_{r=1}^{n} k_r\psi_r \mp \psi_j\right)}{a_j\left(k_0\nu + \sum_{r=1}^{n} k_r\omega_r \pm \omega_j\right)}\right] \quad (22)$$

It should be noted that the harmonic terms in f_j^1 whose frequency is ω_j for τ^* contribute virtual work in equation (11), resulting in resonance. These terms, which are secular terms in equation (17), contribute terms in equation (22).

GYROSCOPIC SYSTEM

Consider the gyroscopic system shown in Figure 1. It consists of a rotor, D, mounted on the shaft, which is supported by two bearings, C and O. The rigidity of the upper bearing, C, is only assumed to be nonlinear with respect to angle β. This assumption is made to simplify the analysis, since the resonance phenomena which will be discussed will be present even if bearing C is also nonlinear with respect to angle α. However, in this case, the analysis will be much more involved. The rotor D is assumed to be unbalanced statically and dynamically.

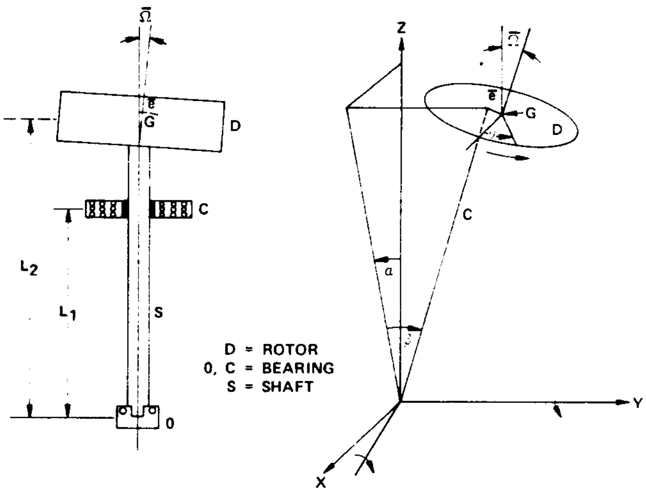

Figure 1. Schematic Representation of a Gyroscopic System Consisting of a Disc Mounted on the Shaft

The differential equations of motion of the rotor D are:

$$I_1\ddot{\alpha} + I_p\dot{\theta}\dot{\beta} + b_1\alpha + \bar{c}\dot{\alpha}$$
$$= [(I_p - I)\,\bar{\Omega} + ML_2\bar{e}][-\dot{\theta}^2 \sin\theta + \ddot{\theta}\cos\theta]$$
$$- \bar{e}P\sin\theta - I_p\ddot{\theta}\beta$$
$$I_1\ddot{\beta} - I_p\dot{\theta}\dot{\alpha} + b_1\beta + \bar{b}_2\beta^2 + \bar{b}_3\beta^3 + \bar{c}\dot{\beta}$$
$$= [(I_p - I)\,\bar{\Omega} + ML_2\bar{e}][\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta]$$
$$- \bar{e}P\cos\theta + I_p\ddot{\theta}\alpha \qquad (23)$$

where

$$I_1 = I + ML_2^2$$
$$b_1 = bL_1^2 - PL_2$$

The bared terms are small and of the order ε. By defining

$$\bar{b}_2 = \varepsilon b_2$$
$$\bar{b}_3 = \varepsilon b_3$$
$$\bar{c} = \varepsilon c$$
$$\bar{e} = \varepsilon e$$
$$\dot{\theta} = \nu(\tau)$$
$$\ddot{\theta} = \varepsilon\,\frac{\partial\nu(\tau)}{\partial\tau}$$
$$\bar{\Omega} = \varepsilon\Omega \qquad (24)$$

substituting equation (24) into equation (23), and neglecting terms of a higher order of ε than unity, we obtain

$$\ddot{\alpha} + \frac{I_p\nu}{I_1}\,\dot{\beta} + K\alpha = \varepsilon\Big[-\delta\dot{\alpha} - F_1\sin\theta - \frac{I_p(\partial\nu/\partial\tau)}{I_1}\,\beta\Big] \qquad (25a)$$

$$\ddot{\beta} - \frac{I_p\nu}{I_1}\,\dot{\alpha} + K\beta = \varepsilon\Big[-\delta\dot{\beta} - K_1\beta^2 - K_2\beta^3 + F_2\cos\theta + \frac{I_p(\partial\nu/\partial\tau)}{I_1}\,\alpha\Big] \qquad (25b)$$

where

$$K = \frac{b_1}{I_1}$$
$$K_1 = \frac{b_2}{I_1}$$
$$K_2 = \frac{b_3}{I_1}$$
$$\delta = \frac{c}{I_1}$$
$$F_1 = \frac{\{[(I_p - I)\,\Omega + ML_2e]\,\nu^2 + eP\}}{I_1}$$
$$F_2 = \frac{\{[(I_p - I)\,\Omega + ML_2e]\,\nu^2 - eP\}}{I_1} \qquad (26)$$

Normalization

Let the solution of equation (25) be in the following form

$$\alpha = \sum_{j=1}^{2} C_jY_j$$
$$\beta = \sum_{j=1}^{2} x_j \qquad (27)$$

where Y_j are indefinite integrals of x_j. If equation (25) is unperturbed, i.e., if $\varepsilon = 0$, x_j is assumed to be a harmonic function of frequency ω_j. Differentiating equation (25a) with respect to t and substituting α and β from equation (27) into the resulting equation and equation (25b) results in the following characteristic determinant:

$$\begin{vmatrix} K - \omega^2 & -\frac{I_p\nu}{I_1}\,\omega^2 \\ -\frac{I_p\nu}{I_1} & K - \omega^2 \end{vmatrix} = 0 \qquad (28)$$

Denoting the roots of equation (28) as

$$\omega = \pm\,\omega_j$$
$$j = 1,\ 2 \qquad (29a)$$

we obtain

$$\omega_1 = \frac{1}{2}\left[\frac{I_p}{I_1}\,\nu - \left(\frac{I_p^2}{I_1^2}\,\nu^2 + 4K\right)^{1/2}\right]$$
$$\omega_2 = \frac{1}{2}\left[\frac{I_p}{I_1}\,\nu + \left(\frac{I_p^2\nu^2}{I_1^2} + 4K\right)^{1/2}\right]$$
$$C_j = -\omega_j$$
$$j = 1,\ 2 \qquad (29b)$$

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Here the modes corresponding to the positive sign in equation (29a) are considered. ω_1, which is negative, represents an inverse precession, and ω_2, which is positive, represents a direct precession. By differentiating equation (25a) with respect to t, substituting α and β from equation (27) into the resulting equation and equation (25b), and solving for $\ddot{x}_j + \omega_j^2 x_j$, we obtain

$$\ddot{x}_j + \omega_j^2 x_j = \varepsilon\Big[-\sum_{i=1}^{2}\lambda_{ji}\dot{x}_i - \sum_{i=1}^{2}\sum_{k=1}^{2} C_{j2}x_i x_k - \sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{\ell=1}^{2} C_{j3}x_i x_k x_\ell + P_j \cos\theta\Big] \qquad j = 1,\ 2 \tag{30}$$

where

$$\lambda_{ji} = (-1)^j\left[\frac{\omega_1\omega_2 I_p}{I_1}\left(\frac{\partial\nu}{\partial\tau}\right) + \delta K\omega_j - \delta\omega_1\omega_2\omega_i - \frac{K\omega_j I_p}{I_1\omega_i}\left(\frac{\partial\nu}{\partial\tau}\right)\right] \Big/ [K(\omega_2 - \omega_1)]$$

$$C_{j2} = (-1)^j\,\frac{\omega_j K_1}{\omega_2 - \omega_1}$$

$$C_{j3} = (-1)^j\,\frac{\omega_j K_2}{\omega_2 - \omega_1}$$

$$P_j = (-1)^j - \frac{\omega_1\omega_2\nu F_1 + K\omega_j F_2}{K(\omega_2 - \omega_1)} \tag{31}$$

Asymptotic Solution

Let us assume that $\nu \neq \omega_1$ or ω_2 for any $\tau^* \varepsilon [0,L]$, i.e., that there is no main resonance. In this case, x_j can be represented as follows:

$$x_j = X_j + A_j(\tau)\cos\theta \tag{32}$$

where $A_j \cos\theta$ is the forced vibration of the linear system due to P_j, and A_j is given by

$$A_j = \frac{\varepsilon P_j}{(\omega_j^2 - \nu^2)} \tag{33}$$

Substituting x_j from equation (32) and A_j from equation (33) into equation (30) and including terms up to the order of ε results in

$$\begin{aligned}\ddot{X}_j + \omega_j^2 X_j = \varepsilon[&g_j^1 + g_j^2\cos\theta + g_j^3\sin\theta \\ &+ g_j^4\cos 2\theta + g_j^5\cos 3\theta \\ &+ g_{j1}^6\dot{X}_1 + g_{j2}^6\dot{X}_2 + (g_{j1}^1 \\ &+ g_{j1}^2\cos\theta + g_{j1}^3\cos 2\theta) \\ &\times (X_1 + X_2) + (g_{j2}^1 \\ &+ g_{j2}^2\cos\theta)(X_1 + X_2)^2 \\ &+ g_{j3}(X_1 + X_3)^3]\end{aligned} \tag{34}$$

where

$$\begin{aligned}
g_j^1 &= -\frac{1}{2}C_{j2}(A_1 + A_2)^2 \\
g_j^2 &= -\frac{3}{4}C_{j3}(A_1 + A_2)^2 \\
g_j^3 &= \nu(\lambda_{j1}A_1 + \lambda_{j2}A_2) + 2\frac{\partial A_j}{\partial\tau}\nu + A_j\frac{\partial\nu}{\partial\tau} \\
g_j^4 &= -\frac{1}{2}C_{j2}(A_1 + A_2)^2 \\
g_j^5 &= -\frac{1}{4}C_{j3}(A_1 + A_2)^2 \\
g_{j1}^6 &= -\lambda_{ji} \\
g_{j1}^1 &= -\frac{3}{2}C_{j3}(A_1 + A_2)^2 \\
g_{j1}^3 &= -\frac{3}{2}C_{j3}(A_1 + A_2)^2 \\
g_{j2}^1 &= -C_{j2} \\
g_{j2}^2 &= -3C_{j3}(A_1 + A_2) \\
g_{j3} &= -C_{j3} \\
g_{j1}^2 &= -2C_{j2}(A_1 + A_2)
\end{aligned} \tag{35}$$

Equation (34) is a special case of equation (1) with n = 2. Confining our analysis to the first asymptotic approximation, from equation (9) we obtain

$$\begin{aligned}
f_j^1 &= f_j(\tau,\theta,X_{10},X_{20},\dot{X}_{10},\dot{X}_{20}) \\
&= g_j^1 + g_j^2\cos\theta + g_j^3\sin\theta \\
&\quad + g_j^4\cos 2\theta + g_j^5\cos 3\theta \\
&\quad - g_{j1}^6 a_1\omega_1\sin\psi_1 - g_{j2}^6 a_2\omega_2\sin\psi_2 \\
&\quad + (g_{j1}^1 + g_{j1}^2\cos\theta + g_{j1}^3\cos 2\theta) \\
&\quad \times (a_1\cos\psi_1 + a_2\cos\psi_2) \\
&\quad + (g_{j2}^1 + g_{j2}^2\cos\theta) \\
&\quad \times (a_1\cos\psi_1 + a_2\cos\psi_2)^2 \\
&\quad + g_{j3}(a_1\cos\psi_1 + a_2\cos\psi_2)^3
\end{aligned} \tag{36}$$

Combination Differential Resonance $\nu = -2\omega_1 + \omega_2$

Assume that

$$\nu(\tau^*) = \omega_2(\tau^*) - 2\omega_1(\tau^*) \tag{37}$$

for some time τ^*. The terms in f_1^1 and f_2^1 which cause the resonance expressed by equation (37) are 1/2 $[g_{12}^2 a_1 a_2 \cos(\theta + \psi_1 - \psi_2)]$ for the resonance relationship $\nu + \omega_1 - \omega_2 = -\omega_1$, and $1/4[g_{22}^2 a_1^2 \cos(\theta + 2\psi_1)]$ for the resonance relationship $\nu + 2\omega_1 = \omega_2$, respectively. Equation (22) can be used to obtain the following nonstationary solution for the resonance relationship $\nu = \omega_2 - 2\omega_1$:

$$\dot{a}_1 = \varepsilon\left[\frac{1}{2} g^6_{11}a_1 - \frac{1}{2} a_1 \frac{(\partial\omega_1/\partial\tau)}{\omega_1} + \frac{1}{2} g^2_{12}a_1a_2 \frac{\sin(\theta + 2\psi_1 - \psi_2)}{\nu - \omega_2}\right] \quad (38a)$$

$$\dot{\psi}_1 = \omega_1 + \varepsilon\left[-\left(\frac{g^1_{11}}{2\omega_1} + 3g_{13}\frac{a_1^2}{8\omega_1} + 3g_{13}\frac{a_2^2}{4\omega_1}\right) + \frac{1}{2} g^2_{12}a_2 \frac{\cos(\theta + 2\psi_1 - \psi_2)}{\nu - \omega_2}\right] \quad (38b)$$

$$\dot{a}_2 = \varepsilon\left[\frac{1}{2} g^6_{22}a_2 - \frac{1}{2} a_2 \frac{(\partial\omega_2/\partial\tau)}{\omega_2} + \frac{1}{4} g^2_{22}a_1^2 \frac{\sin(\theta + 2\psi_1 - \psi_2)}{\nu + 2\omega_1 + \omega_2}\right] \quad (38c)$$

$$\dot{\psi}_2 = \omega_2 + \varepsilon\left[-\left(\frac{g^1_{21}}{2\omega_2} + 3g_{23}\frac{a_2^2}{8\omega_2} + 3g_{23}\frac{a_1^2}{4\omega_2}\right) - g^2_{22}a_1^2 \frac{\cos(\theta + 2\psi_1 - \psi_2)}{4a_2(\nu + 2\omega_1 + \omega_2)}\right] \quad (38d)$$

In the stationary mode, amplitudes a_1 and a_2 are constant; i.e.,

$$\dot{a}_1 = \dot{a}_2 = 0 \quad (39)$$

For the resonance region, using equations (38) and (39), we obtain

$$\frac{a_1^2}{a_2^2} = -2\,\frac{g^6_{22}g^2_{12}\omega_2}{g^6_{11}g^2_{22}\omega_1} \quad (40a)$$

$$\nu = \omega_2 - 2\omega_1 + \varepsilon\left\{\frac{g^1_{11}}{\omega_1} - \frac{g^1_{21}}{2\omega_2} + a_1^2\left[\frac{3g_{13}}{4\omega_1} - \frac{3g_{23}}{4\omega_2} - \left(\frac{3g_{13}}{2\omega_1} - \frac{3g_{23}}{8\omega_2}\right)\left(\frac{g^6_{11}g^2_{22}\omega_1}{2g^6_{22}g^2_{12}\omega_2}\right)\right] \mp \left[-\frac{g^2_{22}g^2_{12}a_1^2}{32\omega_1\omega_2} - \frac{g^6_{22}g^6_{11}}{4}\right] \times \left[2\left(\frac{g^6_{11}}{g^6_{22}}\right)^{1/2} + \left(\frac{g^6_{22}}{g^6_{11}}\right)^{1/2}\right]\right\} \quad (40b)$$

From the Routh-Hurwitz stability criteria, the stability conditions are

$$\pm\frac{\partial\nu}{\partial a_j} > 0, \qquad j = 1, 2 \quad (41)$$

and the stationary amplitude a_1 should be greater than a_1^* which is the solution of the following equation:

$$-\frac{1}{2}(g^6_{11} + g^6_{22}) \times \left[\frac{g^6_{11}g^6_{22}}{4} + \left(-\frac{g^2_{22}g^2_{12}a_1^2}{32\omega_1\omega_2} - \frac{g^6_{22}g^6_{11}}{4}\right) \times \left(\frac{g^6_{22}}{g^6_{11}} - 4\right)\right] + g^2_{22}g^2_{12}a_1^2\left(\frac{2g^6_{11} + g^6_{22}}{64\omega_1\omega_2}\right) \mp 2a_1^2\left(-\frac{g^2_{22}g^2_{12}a_1^2}{32\omega_1\omega_2} - \frac{g^6_{22}g^6_{11}}{4}\right)^{1/2}\left(\frac{4\omega_1}{g^2_{12}}\right)\left(\frac{g^6_{11}}{g^6_{22}}\right)^{1/2} \times \left(-\frac{3g^6_{22}g^2_{22}g_{13}}{16\omega_1\omega_3} + \frac{3g^6_{22}g^2_{22}g_{23}}{64\omega_2^2} + \frac{3g^6_{11}g^2_{12}g_{13}}{32\omega_1^2} - \frac{3g^6_{11}g^2_{12}g_{23}}{32\omega_1\omega_2}\right) = 0 \quad (42)$$

Combination Differential Resonance $\nu = \omega_2 - \omega_1$

Assume that

$$\nu(\tau^*) = \omega_2(\tau^*) - \omega_1(\tau^*) \quad (43)$$

for some time $\tau^*\varepsilon[0,L]$. The terms in f_1^1 and f_2^1 which cause the resonance expressed by equation (43) are 1/2 $g^2_{11}a_2 \cos(\theta - \psi_2)$ for the resonance relationship $\nu - \omega_2 = -\omega_1$, and 1/2 $g^2_{21}a_1 \cos(\theta + \psi_1)$ for the resonance relationship $\nu + \omega_1 = \omega_2$, respectively. Equation (22) can be used to obtain the following nonstationary solution for $\nu = \omega_2 - \omega_1$:

$$\dot{a}_1 = \varepsilon\left[\frac{1}{2} g^6_{11}a_1 - \frac{1}{2} a_1 \frac{(\partial\omega_1/\partial\tau)}{\omega_1} + \frac{1}{2}\,\frac{g^2_{11}a_2 \sin(\theta - \psi_2 + \psi_1)}{\nu - \omega_2 - \omega_1}\right] \quad (44a)$$

$$\dot{\psi}_1 = \omega_1 + \varepsilon\left[-\left(\frac{1}{2}\frac{g^1_{11}}{\omega_1} + \frac{3}{8}\frac{g_{11}a_1^2}{\omega_1} + \frac{3}{4}\frac{g_{13}a_2^2}{\omega_1}\right) + \frac{1}{2} g^2_{11}a_2 \frac{\cos(\theta - \psi_2 + \psi_1)}{a_1(\nu - \omega_2 - \omega_1)}\right] \quad (44b)$$

$$\dot{a}_2 = \varepsilon\left[\frac{1}{2} g^6_{22}a_2 - \frac{1}{2} a_2 \frac{(\partial\omega_2/\partial\tau)}{\omega_2} + \frac{1}{2}\,\frac{g^2_{21}a_1 \sin(\theta - \psi_2 + \psi_1)}{\nu + \omega_1 + \omega_2}\right] \quad (44c)$$

$$\dot{\psi}_2 = \omega_2 + \varepsilon\left[-\left(\frac{1}{2}\frac{g^1_{21}}{\omega_2} + \frac{3}{8} g_{23}\frac{a_2^2}{\omega_2} + \frac{3}{4} g_{23}\frac{a_1^2}{\omega_2}\right) - \frac{1}{2} g^2_{21}a_1 \frac{\cos(\theta - \psi_2 + \psi_1)}{a_2(\nu + \omega_1 + \omega_2)}\right] \quad (44d)$$

The stationary solution is

$$\frac{a_1^2}{a_2^2} = -\frac{g^6_{22}g^2_{11}\omega_2}{g^6_{11}g^2_{21}\omega_1} \quad (45a)$$

$$\nu = \omega_2 - \omega_1 + \varepsilon\left\{\frac{1}{2}\frac{g^1_{11}}{\omega_1} - \frac{1}{2}\frac{g^1_{21}}{\omega_2} + a_1^2\left[\frac{3g_{13}}{8\omega_1} - \frac{3g_{23}}{4\omega_2} - \left(\frac{3g_{13}}{4\omega_1} - \frac{3g_{23}}{8\omega_2}\right) \times \left(\frac{g^6_{11}g^2_{21}\omega_1}{g^6_{22}g^2_{11}\omega_2}\right)\right] \mp \left(-\frac{g^2_{11}g^2_{21}}{16\omega_1\omega_2} - \frac{g^6_{11}g^6_{22}}{4}\right)^{1/2} \times \left[\left(\frac{g^6_{22}}{g^6_{11}}\right)^{1/2} + \left(\frac{g^6_{11}}{g^6_{22}}\right)^{1/2}\right]\right\} \quad (45b)$$

and the stability conditions are

$$\pm\frac{\partial\nu}{\partial a_j} > 0, \qquad j = 1, 2 \quad (46)$$

Numerical Results

The following parameters have been used for numerical calculation:

$$K = 352$$
$$K_1 = 6.25$$
$$K_2 = 9.4$$

$$\frac{I_p}{I_1} = 0.0625$$

$$F_1 = 0.128\nu^2 + 0.375$$

$$F_2 = 0.128\nu^2 - 0.375$$

The nonstationary responses are obtained by numerically integrating the nonstationary solutions. It should be noted that the natural frequencies and amplitude of exciting force are functions of ν; i.e., they are time dependent. The stationary solutions of the system and the nonstationary solutions for various functions of ν are plotted for the combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ in Figures 2 and 3 and Figures 4 and 5, respectively. It is obvious from the nonstationary response that the rate of the frequency of perturbation, ν, plays a significant role in the modification of the nonstationary response. The nonstationary response may be shifted from one stable solution to another by changing the rate of variation of ν.

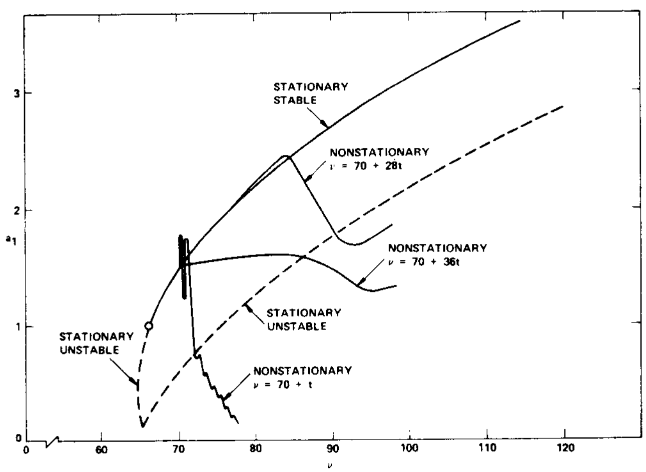

Figure 2. Nonstationary Response for a Combination Differential Resonance, $\nu = \omega_2 - 2\omega_1$, for Linearly Increasing Frequency of Perturbation

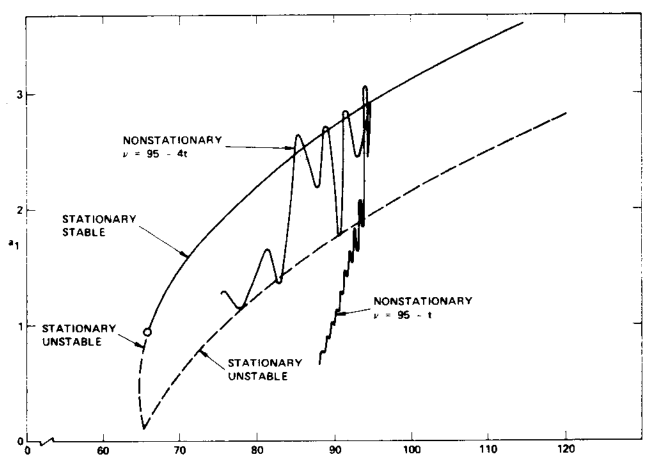

Figure 3. Nonstationary Response for a Combination Differential Resonance, $\nu = \omega_2 - 2\omega_1$, for Linearly Decreasing Frequency of Perturbation

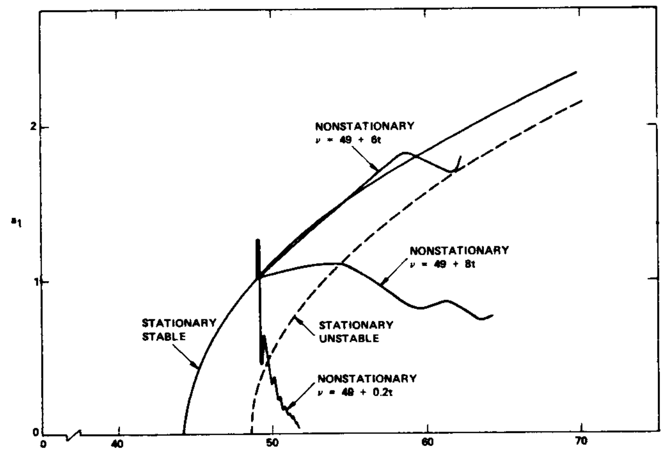

Figure 4. Nonstationary Response for a Combination Differential Resonance, $\nu = \omega_2 - \omega_1$, for Linearly Increasing Frequency of Perturbation

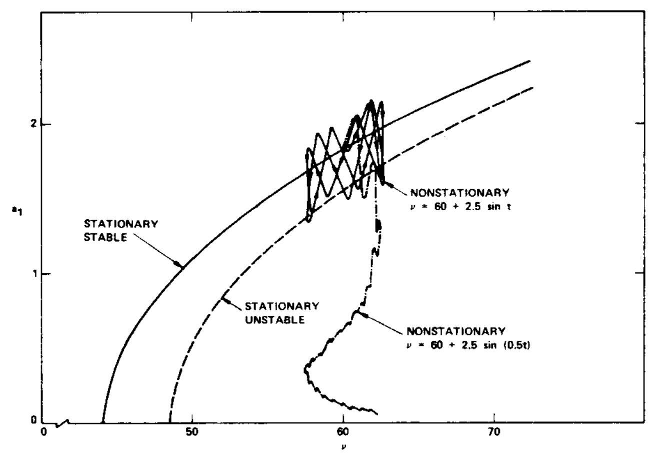

Figure 5. Nonstationary Response for a Combination Differential Resonance, $\nu = \omega_2 - \omega_1$, for Periodically Varying Frequency of Perturbation

CONCLUSIONS

The asymptotic method presented in this paper results in a unified approach for the determination of the resonant response of a nonstationary, nonlinear mechanical system for general resonances, including combination resonances. The resonance conditions can be used to determine the possible resonances in a system. The first asymptotic nonstationary solution can be obtained directly from the general solution, as demonstrated by the calculation of the combination differential resonances $\nu = \omega_2 - 2\omega_1$ and $\nu = \omega_2 - \omega_1$ of the gyroscopic system. The nonstationary responses obtained for various functions of ν indicate that the nonstationary response may shift from one stable mode to another when the rate of variation of the frequency of excitation is changed.

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