

## APPLICATION OF CONSTRAINT DYNAMICS FOR SPACECRAFT MANEUVER

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### *Abstract*

A new approach for spacecraft maneuver based upon constraint dynamics and tracking control laws is presented in this paper. For high precision pointing requirement on spacecraft maneuver, a desired trajectory is prescribed upon which a constraint dynamic system is constructed. A reference model system governed by constraint force is used to build the associated control laws for actual system. Simulation results for single axis slew maneuvers show that this approach can be used for a spacecraft mission with high pointing accuracy requirement.

### **I. Introduction**

Spacecraft maneuver issues have received significant amount of attention during the past few decades. A number of different control law design approaches have been suggested to satisfy various mission objectives. As the recent trends require high precision pointing accuracy, more sophisticated control laws have been considered as being mandatory. One of the key issues in the recent spacecraft maneuver is the embedded flexibility of the spacecraft. Flexibility effect has the dominant effect on the performance of the maneuver and associated control laws. Design of control strategies for spacecraft maneuver with flexibility effect usually raises challenging technical issues.

In this paper, we present flexible spacecraft maneuver problem with prescribed motions which represent the desired mission, i.e., spacecraft trajectory. This approach is essentially close to the so-called inverse dynamics techniques.<sup>1-2</sup> In inverse dynamics approaches, the desired trajectory of a system is determined *a priori*, and the control law is obtained in such a way that the control law, when applied to the actual system, produces the desired trajectory. In robotics area, the inverse dynamics method has been used for various purposes.<sup>1-2</sup> The nature of the inverse dynamics solution is essentially open loop and creates the potential

stability problem. For spacecraft, especially flexible spacecraft maneuver, this approach can be applied to a situation where the accurate pointing maneuver is demanded. For instance, for an on-board antenna or some other parts of spacecraft to follow a prescribed path, the inverse dynamics approach can be used more effectively than typical forward dynamics method.

For a highly flexible spacecraft, the pointing accuracy is limited by the inherent flexibility of the structure. If the prescribed path is generated in such a way that the flexibility effect is minimized over the maneuver, then the control law corresponding to the reference maneuver produces a smooth trajectory with less flexibility effect.

In this study, a prescribed path is introduced in the form of constraint dynamics in conjunction with the Lagrange multiplier. The constraint equation is, in general, a holonomic form such as linear displacement and velocity constraints. The constraint system used as a model system satisfies the constraint equations exactly. The input to the model system is a constraint force developed from constraint equation and the Lagrange multiplier. Two different approaches are investigated to develop control laws based upon the model system. In the first case, direct matching between constraint force and actual control force is sought in an effort to produce actual control input. In other words, the constraint force which is the input to the model system is set equal to the actual desired control force. The solution is open loop, and obtained in the least square sense usually when the number of states are greater than number of control inputs. In the second case, the model system generated by constraint equation is used to develop a tracking control law. The control law is constructed in such a way that the error energy between model system and actual physical system is decreased. This approach is more attractive than the first approach in the sense that the control law maintains stability of the system.

This paper consists of two main parts. In the first part, a flexible spacecraft model is introduced together with the reference system generated. Analytical solution for the constraint dynamics is sought as an analytical expression in terms of the Lagrange multiplier. In the second part, control laws are designed based upon the constraint dynamics system created in the first part.

### **II. Problem Definition and Model System**

The physical model used in this study is presented

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in Fig. 1. It represents a spacecraft with a rigid main body and two identical flexible structures attached to the main body, and each flexible structure has an antenna at the tip. For simplicity of analysis, the system is modified into the model in Fig. 2, where the truss structures are modeled as beams with tip masses attached. The flexible beams are under bending vibration and assumed to satisfy Euler-Bernoulli beam assumptions. The only control device is a reaction wheel located at the center body. As a particular case, the motion of the whole structure is restricted to a plane motion creating couplings between the center body motion and vibration of the appendages. This model is similar to the one that has been used in the previous studies.<sup>3-4</sup>

For a single axis slew maneuver, the governing linearized ordinary/partial differential equations of motion are given in the form<sup>3-4</sup>

$$I_c \ddot{\theta} + 2 \left[ \int_{l_0}^l \rho x (\ddot{y} + x \ddot{\theta}) dx + m_t l (\ddot{\theta} + \ddot{y}|_l) + I_t (\ddot{\theta} + \ddot{y}'|_l) \right] = u$$

$$\rho (\ddot{y} + x \ddot{\theta}) + EI \frac{\partial^4 y}{\partial x^4} = 0 \quad (1)$$

with the boundary conditions

$$y(x, t) = y'(x, t) = 0 \quad \text{at } x = l_0$$

$$EI y'' = -I_t (\ddot{\theta} + \ddot{y}'), \quad EI y''' = m_t (l \ddot{\theta} + \ddot{y}), \quad \text{at } x = l$$

where  $(\ )' \equiv \frac{\partial}{\partial x}(\ )$ ,  $I_c$  is moment of inertia of center hub,  $m_t$  tip mass,  $I_t$  moment of inertia of tip mass,  $\rho$  linear mass density of the beam,  $EI$  elastic rigidity of the beam,  $u$  applied torque,  $l_0$  center body radius,  $l$  total undeformed length of the structure, and  $\theta$  is center body angle. In addition, as shown in Fig. 2, the two beams are assumed to be in anti-symmetric deformation so that there is no net translational force generated at the center body.

Next, for the above structural system, a desired trajectory is prescribed. As a particular case, it is assumed that the motion of the tip is desired to follow a certain path with respect to time. This statement, however, covers a rather general class of pointing problems by prescribing trajectories of some parts of the spacecraft. In our case, the tip motion is prescribed by the following equation.

$$y(l, t) + l\theta(t) = c(t) \quad (2)$$

where  $c(t)$  is a trajectory generated in such a way that the resulting motion is as smooth as possible. In this study,  $c(t)$  is generated by piecewise continuous smooth polynomials in time. This is motivated by a near-minimum-time rigid body maneuver in the recent studies by Junkins *et. al.*<sup>3-4</sup> Figure 3 shows  $c(t)$ , a

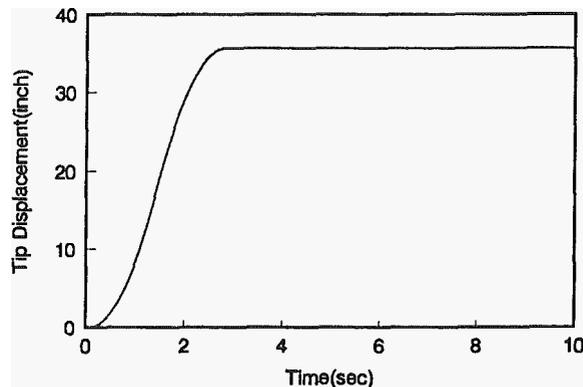


Figure 3 Prescribed tip trajectory

smooth trajectory with insignificant flexibility. Even if  $y(l, t) + l\theta(t)$  does not have physical meaning, it dictates the overall motion, i.e., both rigid and flexible motion of the tip. In the previous studies, the same parameter has 'been used as a reference trajectory.' As mentioned earlier, the model system should be generated first, which satisfies the constraint equation. In order to make use of the standard approach for constraint dynamics, first we discretize the original equation, Eq. (1), into a finite dimensional ordinary differential equations. For finite approximation, the following expansion is introduced

$$y(x, t) = \sum_{i=1}^N \phi_i(x - l_0) \eta_i(t) \quad (3)$$

where  $\phi_i(x)$  is the  $i$ -th shape function which satisfies both geometrical and natural boundary conditions.<sup>5</sup> By applying the Lagrange's equations by taking  $\theta, \eta_1, \eta_2, \dots, \eta_N$  as generalized coordinates with introduction of Eq. (3), we obtain

$$\begin{bmatrix} M_{\theta\theta} & M_{\eta\theta} \\ M_{\eta\theta}^T & M_{\eta\eta} \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{\eta\eta} \end{bmatrix} \begin{Bmatrix} \theta \\ \eta \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad (4)$$

where  $\eta = [\eta_1, \eta_2, \dots, \eta_N]^T$  and  $(\ )'' = \frac{\partial^2}{\partial x^2}(\ )$ . In addition, each sub-matrix is defined as follows

$$M_{\theta\theta} = I_c + 2 \left[ \int_{l_0}^l \rho x^2 dx + I_t + m_t l^2 \right]$$

$$M_{\eta\theta} = \int_{l_0}^l \rho x \phi_i(x - l_0) dx$$

$$M_{\eta\eta} = \int_{l_0}^l \rho \phi_i(x - l_0) \phi_j(x - l_0) dx$$

## 11. Control Law Design

In this section, the model system, i.e., the virtual system subject to constraint force  $\mathbf{a}$  in Eq. (10) is used to design control laws for the actual system. Two different approaches—direct matching and tracking law—are investigated.

### a). Direct matching approach

In the direct matching approach, the model system developed in the previous section is used to find the actual control law so that the actual control input matches the constraint force in the model system. The control action is taken in a manner to generate the constraint force which is evidently related to the desired path constraint. If there exist an exact solution, then control input satisfies the constraint equation exactly. For the actual system in Eq. (6)

$$M\ddot{\mathbf{q}} + K\mathbf{q} = F\mathbf{u}$$

by using Eq. (7), we seek to find a control input  $u$  which matches the constraint force  $\mathbf{a}$  as follows

$$F\mathbf{u} = A^T\lambda$$

Exact solution for the above equation mainly depends on the size of  $F$ . In general,  $F$  is not invertible due to the number of controls usually less than number of states. A possible solution is found in the least square sense as follows

$$\begin{aligned} \mathbf{u} &= F^\dagger A^T \lambda \\ &= F^\dagger A^T \tilde{A}^{-1} A M^{-1} K \mathbf{q}_m + F^\dagger A^T \tilde{A}^{-1} a(t) \\ &\equiv G_1 \mathbf{q}_m + G_2 a(t) \end{aligned} \quad (11)$$

where  $F^\dagger$  is the *generalized inverse* of  $F$  and  $G_1, G_2$  are feedforward gains. As we can see, the control input consists of two parts: one from  $\mathbf{q}_m$  and the other one from  $a(t)$ . Upon substituting Eq. (11) into the actual system, it follows that

$$M\dot{\mathbf{q}} + I\dot{q} = \tilde{F}^\dagger A^T \lambda$$

where  $\tilde{F} = FF^\dagger$ . By eliminating  $\lambda$ , the above equation can be rewritten as

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \tilde{F}A^T \tilde{A}^{-1} A M^{-1} K \mathbf{q}_m + \tilde{F}A^T \tilde{A}^{-1} a(t) \quad (12)$$

The actual system is driven by combination of both  $\mathbf{q}_m$  and  $a(t)$ . As discussed earlier, the model system  $q_m$  is dictated by a new system with a new stiffness matrix as Eq. (10), which excludes the possibility of resonance between model system and actual system. Both Eqs. (10) and (12) are combined together to produce an actual system and the associated model system. For actual implementation of this approach,

$$K_{\eta\eta} = \int_{l_0}^l EI \phi_i''(x - l_0) \phi_j''(x - l_0) dx$$

where  $i, j = 1, 2, \dots, N$ . On the other hand, the constraint equation can be rewritten as

$$Aq = c(t) \quad (5)$$

where

$$\begin{aligned} A &= [l, \phi_1(L), \phi_2(L), \dots, \phi_N(L)] \\ \mathbf{q} &= [\theta, \eta_1, \eta_2, \dots, \eta_N]^T \end{aligned}$$

and  $L = l - l_0$ . For notational simplicity, Eq. (4) is rewritten in the form

$$Mq + I\dot{q} = F\mathbf{u} \quad (6)$$

where  $M$  is a mass matrix,  $K$  is a stiffness matrix, and  $F$  is an input distribution matrix. In order to generate a model system, which satisfies the constraint equation, Eq. (5), we assume a model system being subject to a constraint force. In other words, by excluding the control force, the governing equation for the model system is

$$M\ddot{\mathbf{q}}_m + K\mathbf{q}_m = A^T \lambda \quad (7)$$

where  $\lambda$  is a Lagrange multiplier,  $A^T \lambda$  is a constraint force, and  $(\ )_m$  is introduced to denote states corresponding to the model system. Also, the constraint equation, Eq. (5), can be rewritten as

$$A\mathbf{q}_m = c(t) \quad (8)$$

From Eqs. (7) and (8) the Lagrange multiplier is analytically obtained in the form<sup>6</sup>

$$\lambda = \tilde{A}^{-1} A M^{-1} K \mathbf{q}_m + \tilde{A}^{-1} a(t) \quad (9)$$

where

$$\tilde{A} = A M^{-1} A^T, \quad a(t) = \frac{d^2}{dt^2} c(t)$$

By substituting Eq. (9) into Eq. (7), the new system is obtained as

$$M\ddot{\mathbf{q}}_m + \bar{K}\mathbf{q}_m = A^T \tilde{A}^{-1} a(t) \quad (10)$$

where

$$\bar{K} = [I - A^T \tilde{A}^{-1} A M^{-1}] K$$

Therefore by eliminating the Lagrange multiplier, we obtain a new stiffness matrix,  $\bar{K}$ . The property of the new stiffness matrix is dependent upon the constraint equation,  $A$ . It is worthwhile, as will be shown later, to note that  $q_m$  is governed by the new stiffness matrix ( $\bar{K}$ ). In other words, the natural frequency has been changed due to the introduction of the Lagrange multiplier.

the model system in Eq. (10) should be solved either on-line or off-line basis.

### b). Tracking control law

The possible disadvantage of the previous direct matching method is due to the nature of the open control law with lack of stability guarantee. The model system is built upon zero initial condition assumption which is not perfectly guaranteed in the actual physical system. In this approach, we seek to design a stabilizing control law by Lyapunov approach. A candidate Lyapunov function is chosen as

$$U = \frac{1}{2} \mathbf{e}^T M \mathbf{e} + \frac{1}{2} \mathbf{e}^T K \mathbf{e} + \frac{1}{2} \mathbf{e}^T Q \mathbf{e} \quad (13)$$

where  $\mathbf{e} = \mathbf{q} - \mathbf{q}_m$ , and  $Q$  is a positive definite matrix. Obviously the Lyapunov function  $U$  is positive definite and represents an error energy between the model system and the actual system. Taking time derivative of  $U$  in combination with Eqs. (6) and (7) yields

$$\frac{dU}{dt} = (\dot{\mathbf{q}} - \dot{\mathbf{q}}_m)^T [F\mathbf{u} - A^T \lambda + Q(\mathbf{q} - \mathbf{q}_m)]$$

For stability in the Lyapunov sense, the control law is chosen to satisfy

$$F\mathbf{u} - A^T \lambda + Q(\mathbf{q} - \mathbf{q}_m) = -\bar{Q}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_m)$$

where  $Q$  is a positive definite matrix. The control input  $\mathbf{u}$ , similar to the previous section, is obtained in the least square sense as follows

$$\begin{aligned} \mathbf{u} &= -F^\dagger \bar{Q}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_m) - F^\dagger Q(\mathbf{q} - \mathbf{q}_m) + F^\dagger A^T \lambda \quad (14) \\ &\equiv G_1 \mathbf{q}_m + G_2 a(t) + G_3(\dot{\mathbf{q}}_m - \dot{\mathbf{q}}) + G_4(\mathbf{q} - \mathbf{q}_m) \end{aligned}$$

where  $G_i (i = 1, \dots, 4)$  are feedback/feedforward gains. In comparison with the direct matching law of Eq. (11), the control law consists of both feedback and feedforward terms. The actual closed loop system applied by the control law has the form

$$M\ddot{\mathbf{q}} + \tilde{F}\bar{Q}\dot{\mathbf{q}} + [K + \tilde{F}Q]\mathbf{q} = \tilde{F}\bar{Q}\dot{\mathbf{q}}_m + \tilde{F}Q\mathbf{q}_m + \tilde{F}A^T \lambda \quad (15)$$

where  $\lambda$  should be replaced by the expression in Eq. (9) so that the right hand side of Eq. (15) consists of  $\mathbf{q}_m$  and  $a(t)$ . It should be noted that the above closed loop system has some margin for stability guarantee in the sense that  $\tilde{F}\bar{Q} \geq 0$  in this special case. The stability of model system in Eq. (10), however, is not guaranteed since the modified stiffness matrix ( $\bar{K}$ ) does not introduce any damping effect. The constraint

equation usually makes  $\bar{K}$  marginally stable, and this should be regarded as a potential problem in the actual implementation. Furthermore, it is assumed that the generalized coordinate vector ( $\mathbf{q}$ ) in Eq. (15) can be estimated using a dynamic estimator in real time.

## IV. Application and Simulation Results

Two independent control laws, Eqs. (11) and (14), are applied for the simulation purposes. The feedback/feedforward gains associated with the control laws of Eqs. (11) and (14) are provided in Table 2. The feedforward gains for the center body angle is identically zero which can be explained by the fact that there should be no stiffness effect on the hub motion.

Simulation results based upon the control law of Eq. (11) are provided in Fig. 4. Both identical and different initial conditions between the model system and the actual system are examined. With the same initial conditions, in spite of the errors caused by the least square solutions, the Constraint equation is satisfied with enough accuracy. The applied control input is also presented in Fig. 4. The vibratory motion of the control input is observed and it turns out that the motion is dominated by the first natural frequency of flexible modes. The constraint equation where the motion was defined to be a pure rigid motion produces the vibration in the control input. Also, the error in the least square solution may result in some discrepancy between constraint trajectory and resultant trajectory. On the other hand, when there exist initial condition error in the actual system, the pointing error is not controlled. This is evident from the nature of the control law in Eq. (11), which is open-loop type with no feedback on the error.

Simulation results with the control law of Eq. (14) are also provided in Fig. 5. Similar performance of the control law as the previous control law is shown with satisfactory tracking accuracy. The slight overshoot of the response is caused by the high position feedback gain, in combination with the approximate solution of the generalized inverse, for which the exact solution is not available. As discussed earlier, the control law of Eq. (14) has some advantages over the control law of Eq. (11) in the sense that the control law is a feedback law. As is shown in Fig. 5, the initial error on the actual system is controlled in this case. It is worthwhile to note that the control input fluctuation level is less than that of control law in Eq. (11).

For further verification of the proposed method, a sinusoidal reference motion is tried as.

$$a(t) = N \sin(\omega t), \quad c(t) = \frac{N}{\omega} [t - \sin(\omega t)/\omega] \quad (16)$$

where  $\omega = 10 \text{ rad/sec}$  and  $N$  is a constant. Simulation results with the above reference trajectory applied are

presented in Figs. 6 and 7. Similar trends are observed in comparison with the Figs. 4 and 5. This demonstrates the possibility of applying the proposed method to a general class of reference motions.

### V. Conclusions

Application of constraint dynamics has been made to design control laws which satisfy the constraint equation in least square error sense. The suggested approach provides us with a direct path to creating a model system and designing a tracking control law in conjunction with the model system and the actual system. Simulation results support the useful aspects of the method. It is expected that the approach herein can be extended into a large class of spacecraft maneuver applications including general nonlinear three axis maneuvers. For actual application, however, it requires further study, especially, concerning on-board processor capability and stability of the system with constraint.

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**Table 1 Space structure model parameters**

PARAMETER	SYMBOL	VALUE
Center body radius	$l_0$	3.504 in
Center body inertia	$I_c$	9.061 oz-sec <sup>2</sup> -in
Mass density of beams	$\rho$	0.003 oz-sec <sup>2</sup> /in <sup>2</sup>
Elastic modulus of beams	E	161.6×10 <sup>6</sup> oz/in <sup>2</sup>
Beam thickness	t	0.118 in
Beam height	h	5.94 in
Beam length	$L(r_1 - l_0)$	47.568 in
Tip mass	$m_t$	0.156 oz-sec <sup>2</sup> /in
Tip inertia	$I_t$	0.0018 oz-sec <sup>2</sup> -in

**Table 2 Feedback and feedforward gains**

$G_1(\times 10^5) = [ 0.0000, 0.0029, -0.1055, 0.6119, -1.8871, 4.1547, -7.9143 ]$
$G_2 = 16.458$
$G_3 = [ 1000, 0, 0, 0, 0, 0, 0, 0 ]$
$G_4 = [ 1000, 0, 0, 0, 0, 0, 0, 0 ]$

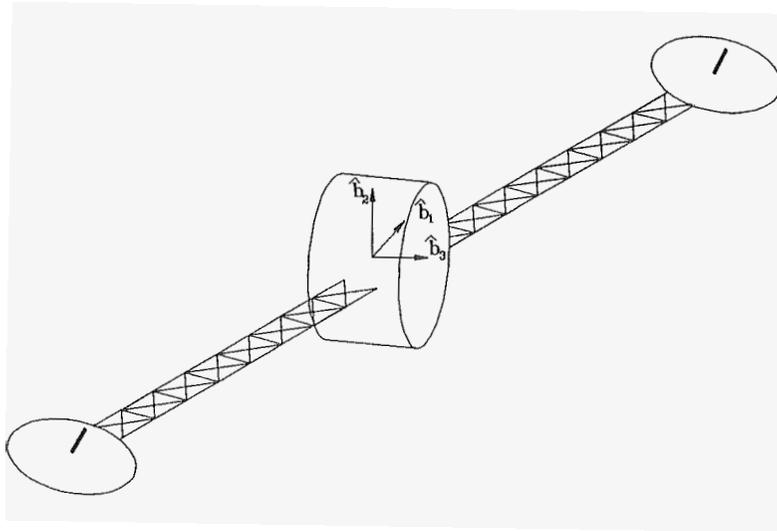


Figure 1 Flexible Spacecraft Configuration

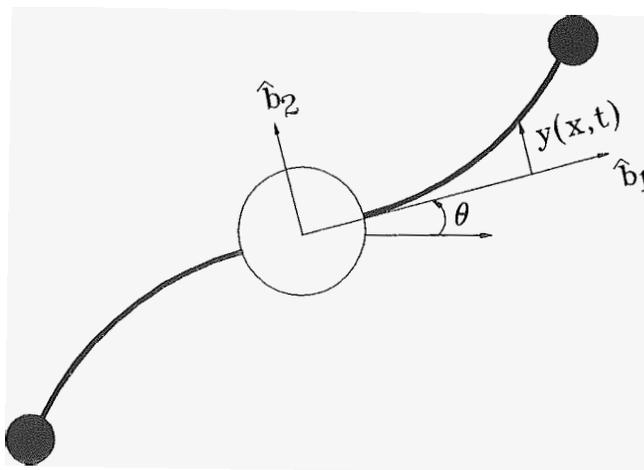


Figure 2 Simplified Model with Unsymmetric Motion

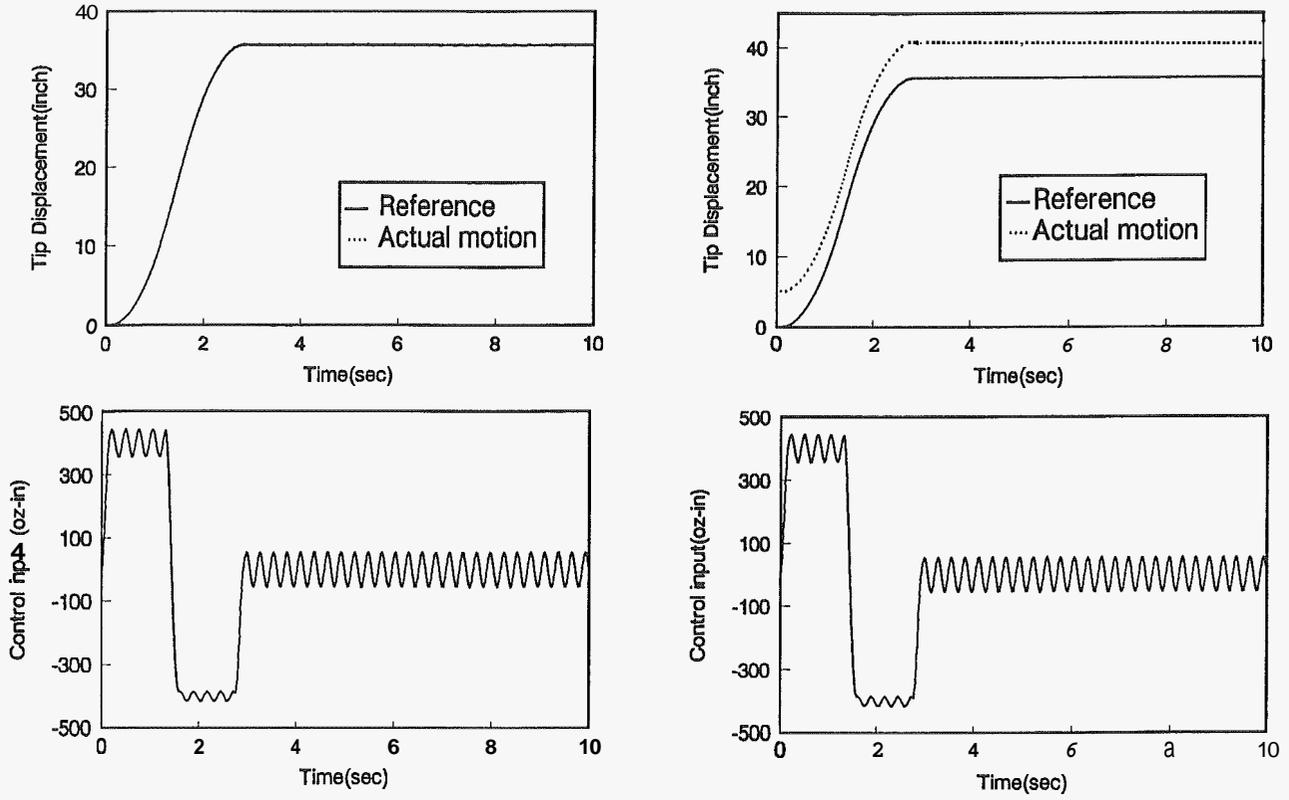


Figure 4 Simulation results with direct matching control law

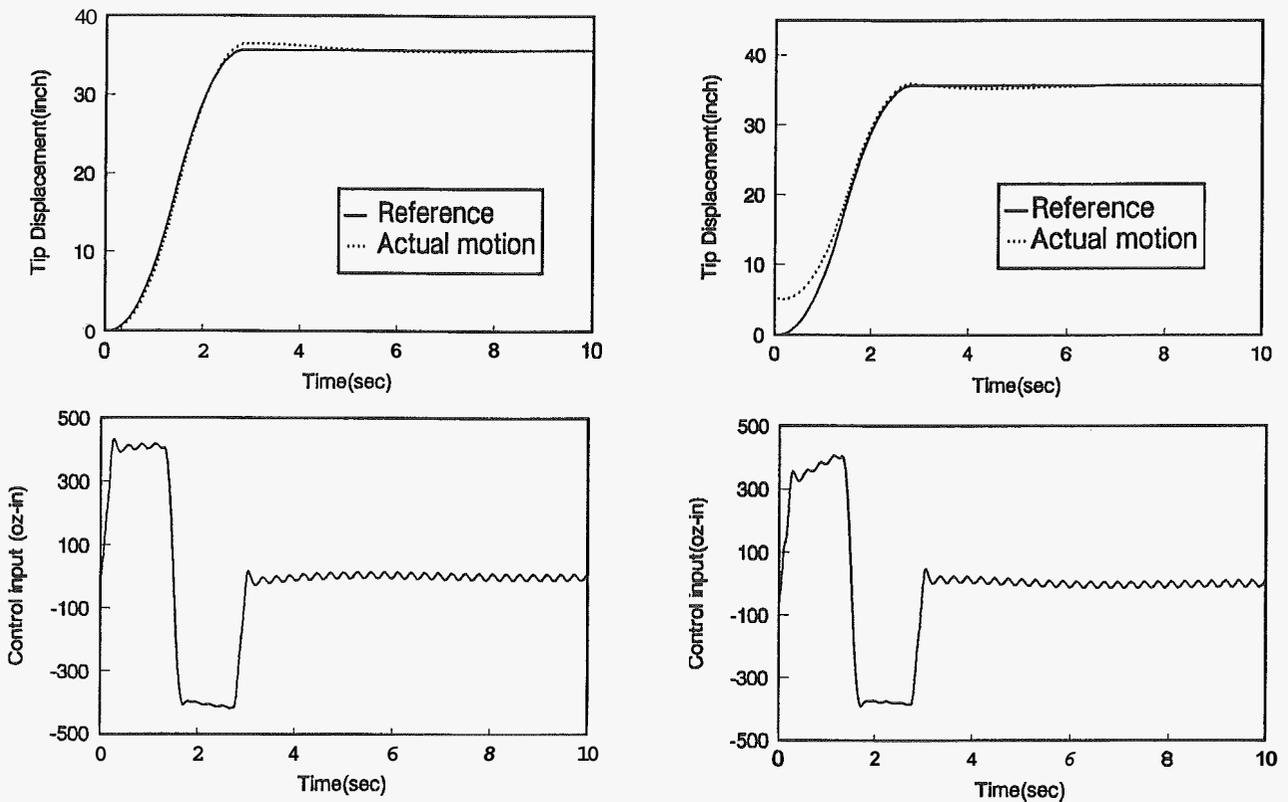
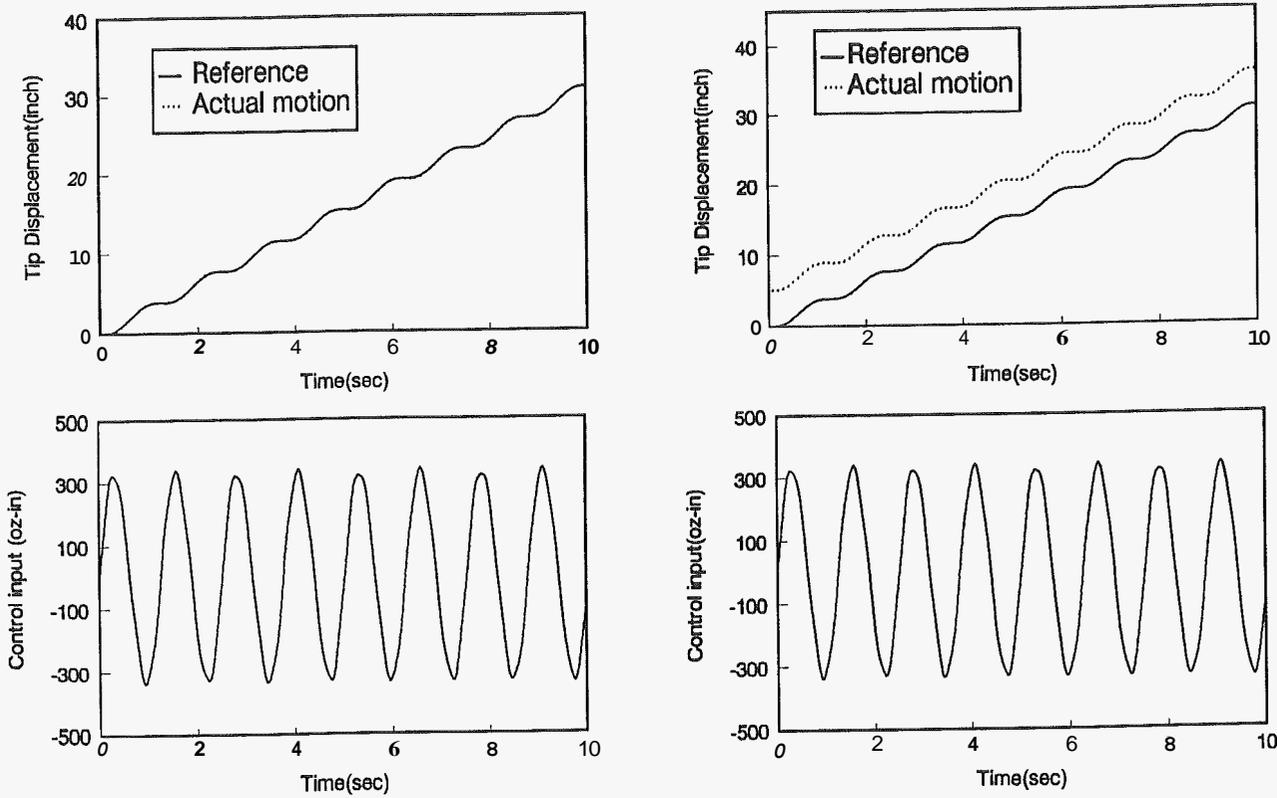
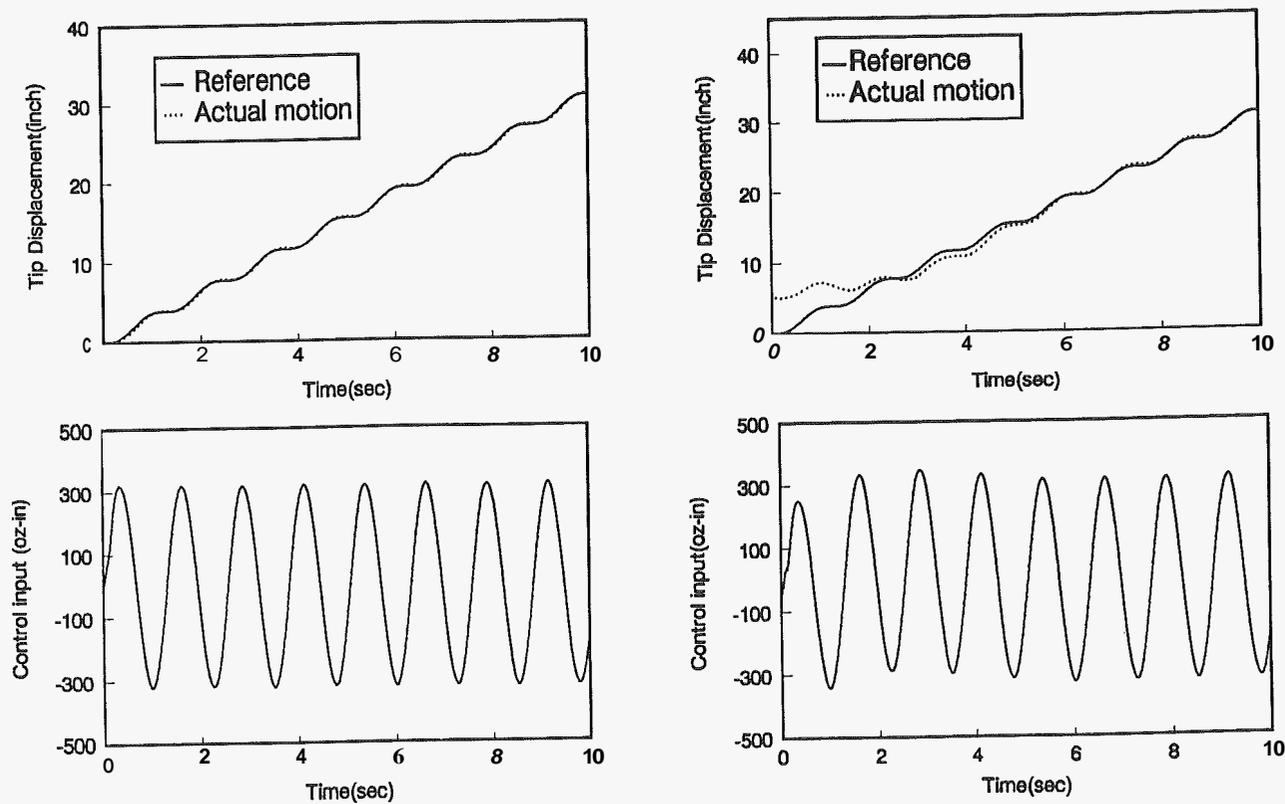


Figure 5 Simulation results with feedback tracking control law



**Figure 6 Simulation results with direct matching control law and sinusoidal reference trajectory**



**Figure 7 Simulation results with feedback tracking control law and sinusoidal reference trajectory**