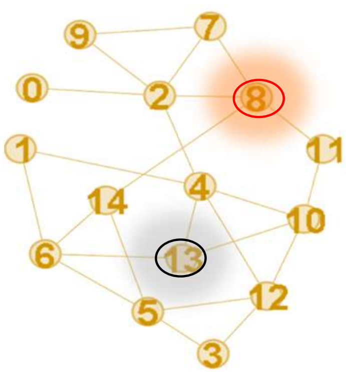
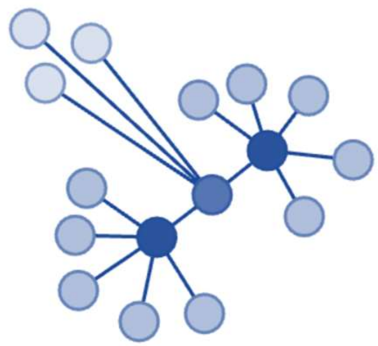
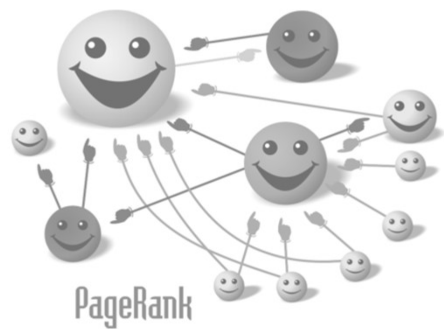
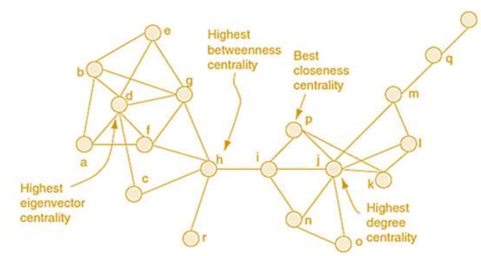
Prof. Ralucca Gera, rgera@nps.edu
Applied Mathematics Department,
Naval Postgraduate School









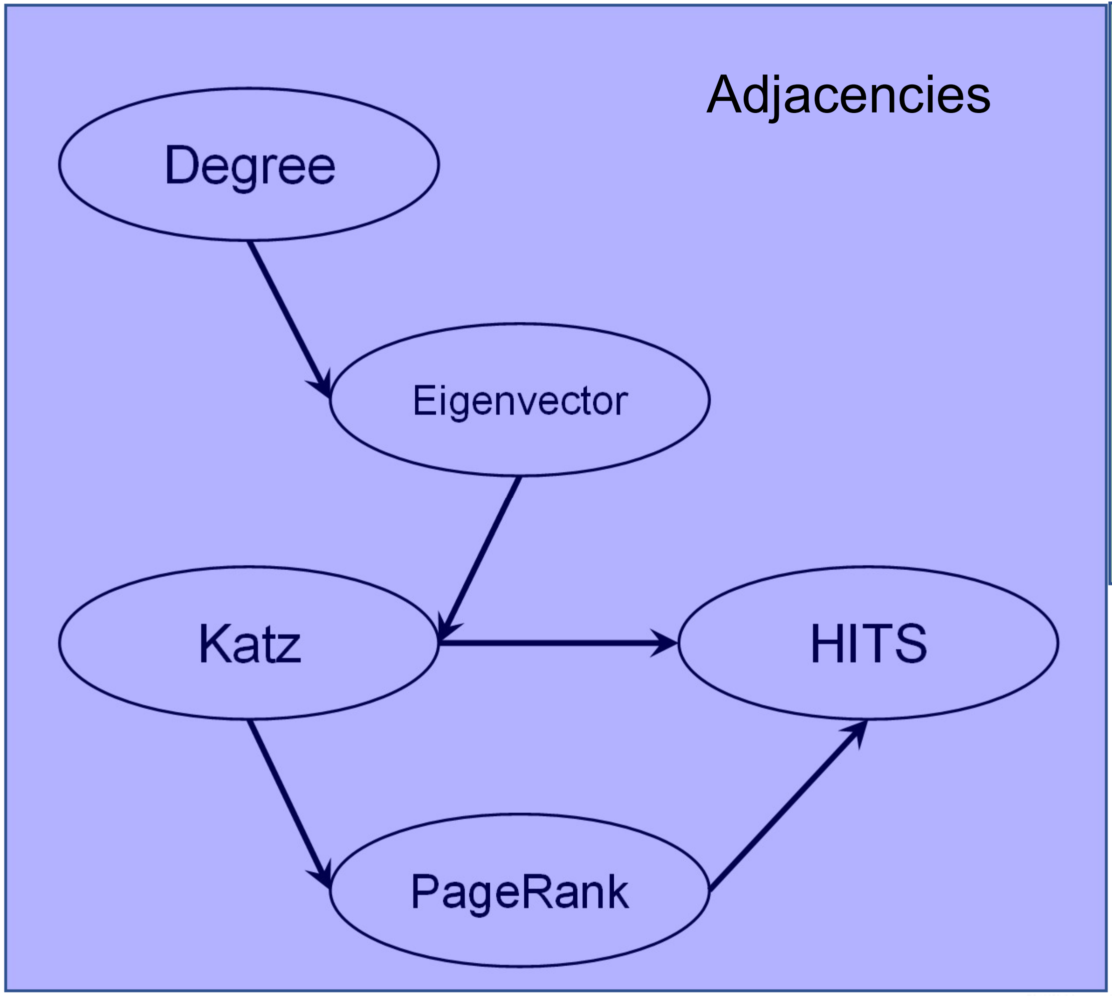
MA4404 Complex Networks

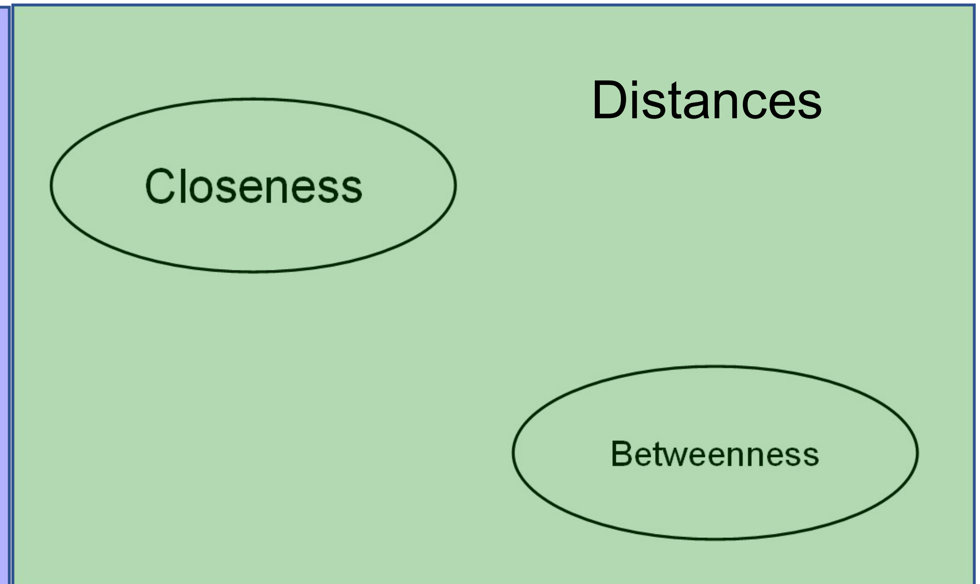
Eigenvector Centrality

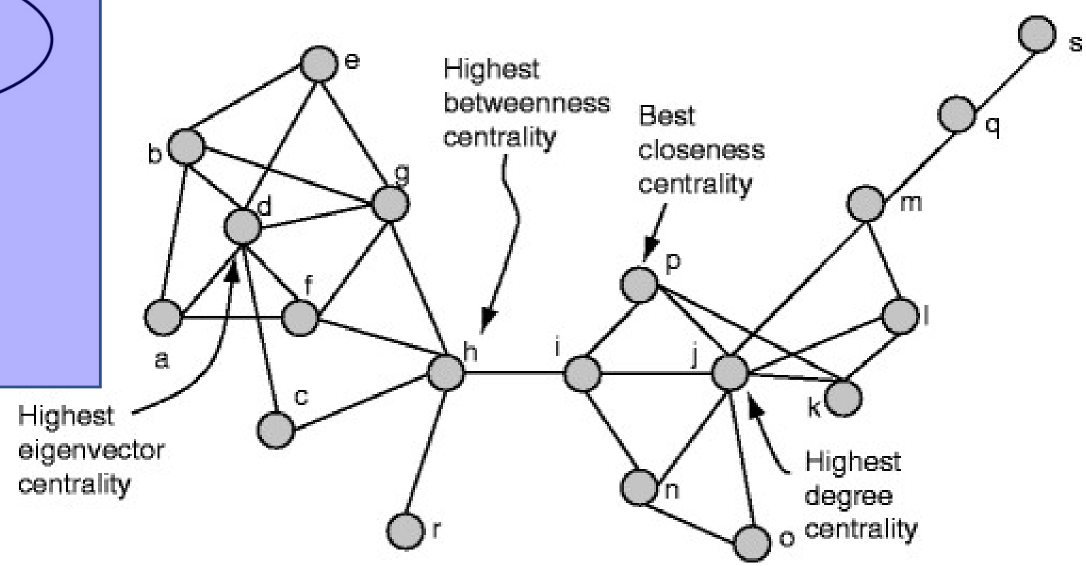
Learning Outcomes

- Compute eigenvector centrality.
- Interpret the meaning of the values of eigenvector centrality.
- Explain why the eigenvector centrality is an extension of degree centrality.

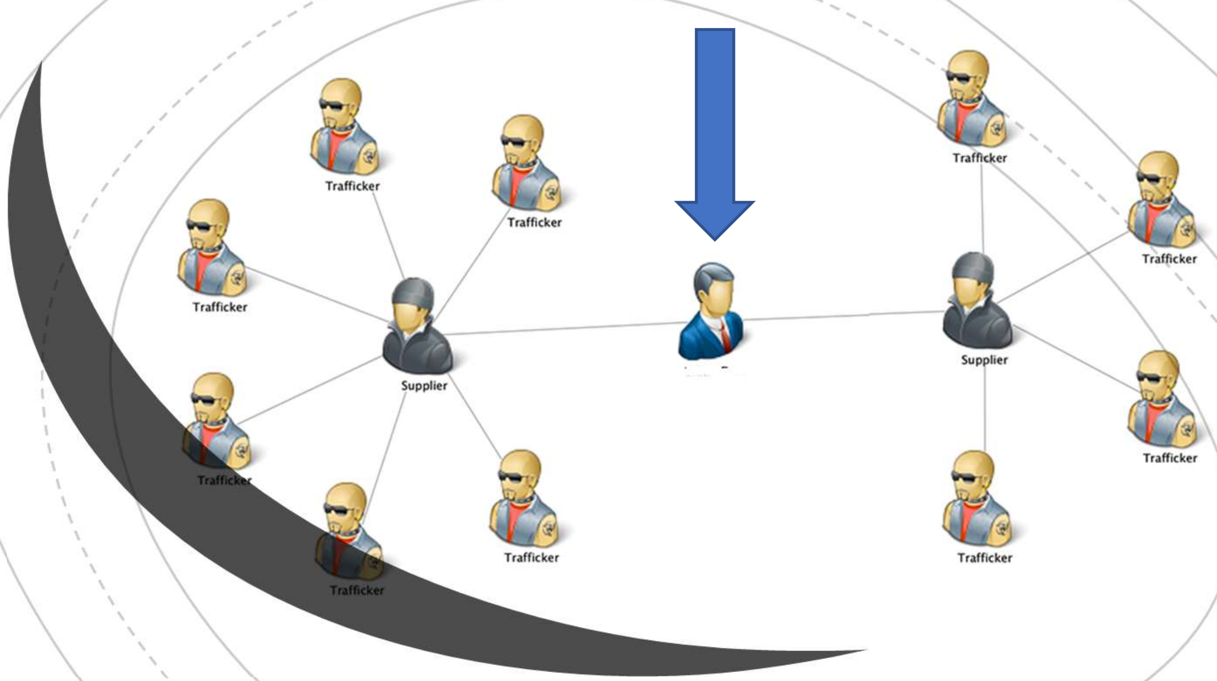
MA4404: Centralities categories







[Discovering Sets of Key Players in Social Networks | SpringerLink]



Eigenvector Centrality

Recall:

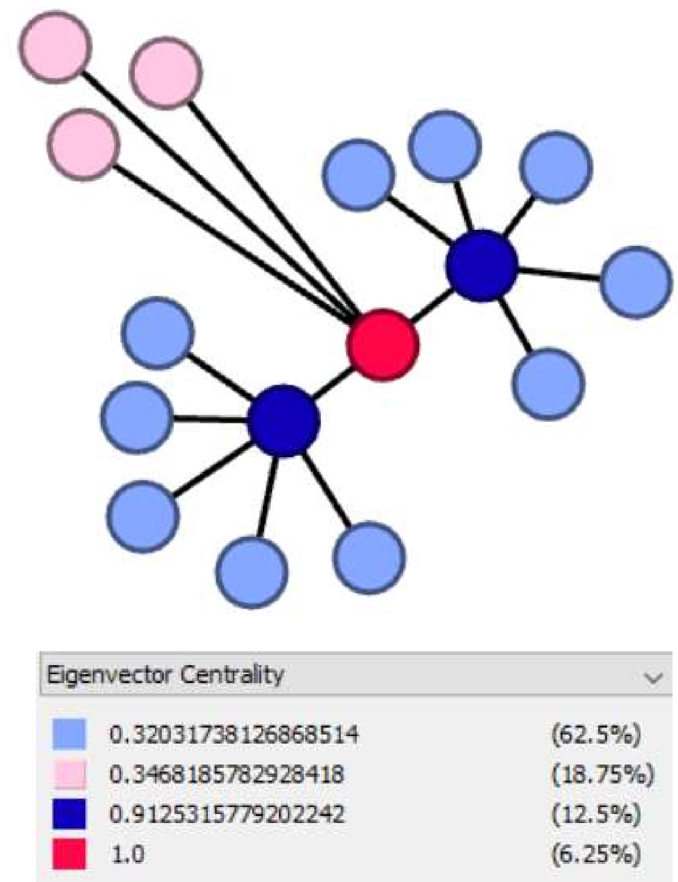
Quality: what makes a node important (central)	**Mathematical Description**	**Appropriate Usage**	**Identification**
Lots of one-hop connections from v	The number of vertices that v influences directly	Local influence matters Small diameter	Degree $\deg(i)$
Lots of one-hop connections from v relative to the size of the graph	The proportion of the vertices that v influences directly	Local influence matters Small diameter	Degree centrality $\mathrm{C_i} = \frac{\deg(i)}{\lvert V(G)\rvert}$
Lots of one-hop connections to high centrality vertices	A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality)	For example when the people you are connected to matter.	HOW? Eigenvector centrality (recursive formula): $C_i \propto \sum_{j \in N(i)} C_j$

Eigenvector Centrality

- A generalization of the degree centrality:
  a weighted degree vector that depends on the centrality of its neighbors (rather than every neighbor having a fixed centrality of 1)

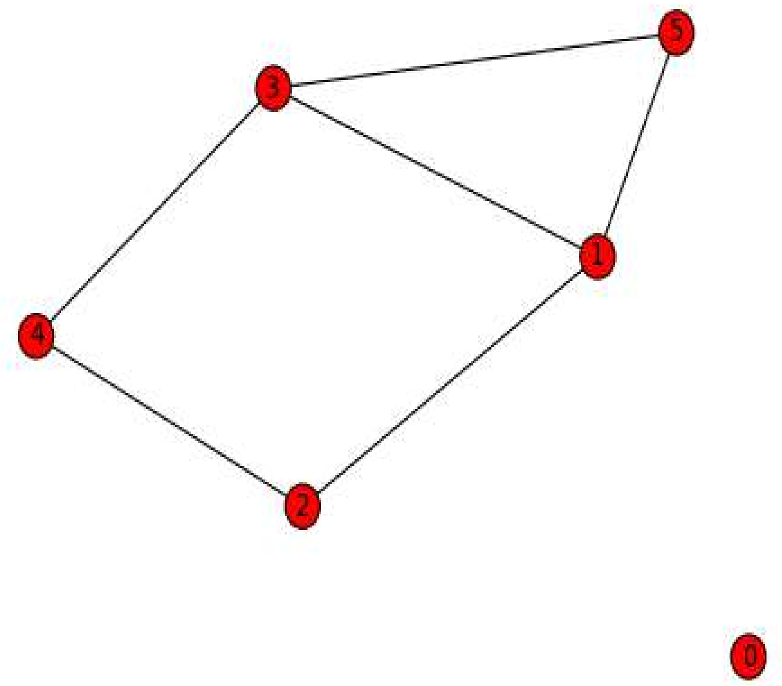
- How do we find it? By finding the largest eigenvalue and its associated eigenvector (leading eigenvector) of the adjacency matrix

- Let’s see why



Example 1 (Eigenvector centrality)

Node i	Eigenvector centrality C_i
0	0
1	0.5298987782873977
2	0.3577513877490464
3	0.5298987782873977
4	0.3577513877490464
5	0.4271328349194304



Notice that deg(5) =deg (1) = deg(3).
Why $C_5 > C_2$ and $C_5 > C_4$?

Computing Eigenvector Centrality

$$x_i(t) = \sum_j A_{ij} x_j(t-1)$$

with the centrality at time t=0 being

$$x_j(0) = 1, \ \forall j$$

Eigenvector Centrality

- Define the centrality ${x'}_i$ of i recursively in terms of the centrality of its neighbors

$$x_i' = \sum_{k \in N(i)} x_k \qquad or \qquad x_i' = \sum_j A_{ij} x_j$$

With initial vertex centrality $x_j = 1,\ \forall j\ (including\ i)$—we'll see why on next slide

- That is equivalent to:

$$x_i(t) = \sum_j A_{ij} x_j(t-1)$$

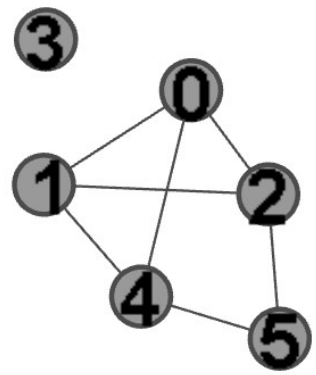
with the centrality at time t=0 being $x_j(0) = 1,\ \forall j$

The centrality of vertices i and j at time t and t-1, respectively

In class: Eigenvector Centrality

Adjacency matrix A for the graph to the right:

$$A=\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



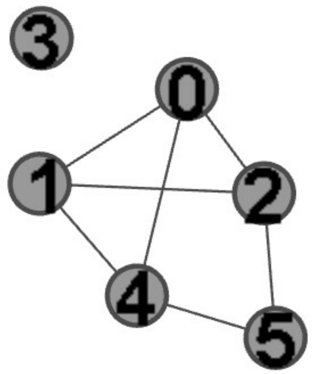
Then the vector $x(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ gives a random surfer's behavior at time t.

Answer the following questions based on the information above

In class activity: Eigenvector Centrality

Q1: Find x(1). What does it represent?

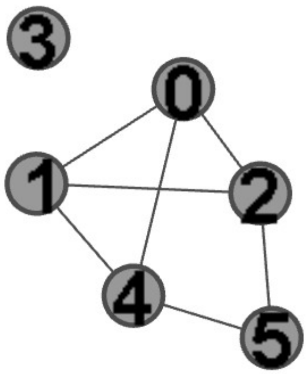
Answer: $x(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (?)$



In class activity: Eigenvector Centrality

Q1: Find x(1). What does it represent?

Answer: $$x(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 3 \\ 2 \end{pmatrix}$$

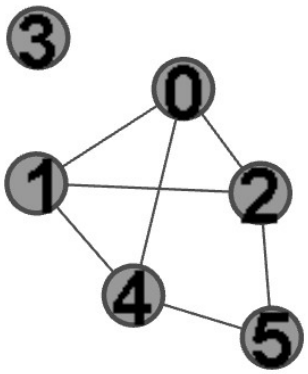


The degree vector

In class activity: Eigenvector Centrality

Q2: Find x(2). What does it represent?

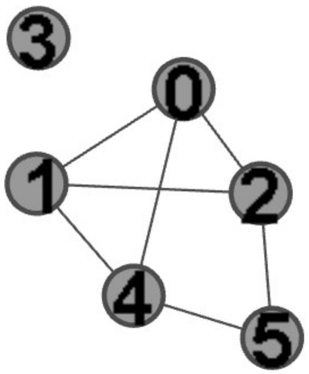
Answer: $x(2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 3 \\ 2 \end{pmatrix} = (?)$



In class activity: Eigenvector Centrality

Q2: Find x(2). What does it represent?

Answer: $$x(2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix}$$

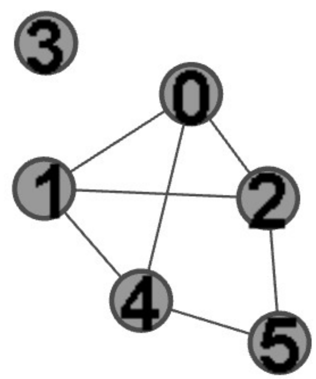


A weighted degree vector (distance 2 or less)

In class activity: Eigenvector Centrality

Q3: Find x(3). What does it represent?

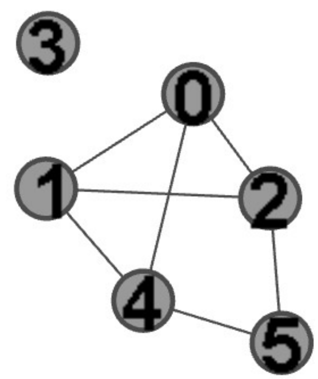
Answer: $x(3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix} = (?)$



In class activity: Eigenvector Centrality

Q3: Find x(3). What does it represent?

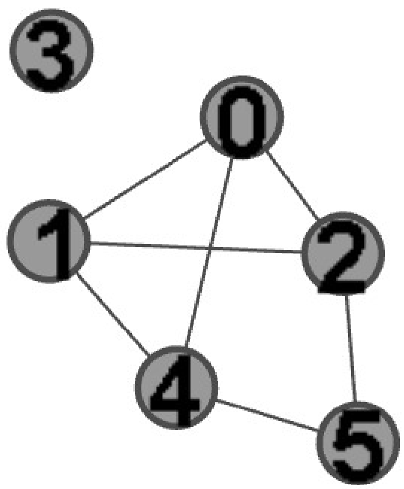
Answer: $x(3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \\ 8 \\ 0 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 25 \\ 25 \\ 24 \\ 0 \\ 24 \\ 16 \end{pmatrix}$



A weighted degree vector (distance 3 or less)

In class: Eigenvector Centrality Results

Node i	Eigenvector centrality C_i
0	0.49122209552166
1	0.49122209552166
2	0.4557991200411896
3	0
4	0.4557991200411896,
5	0.31921157573304415



A normalized weighted degree vector

$$x(\mathrm{t}) = A(A \ldots (A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}))) = A^t x(0), t > 0$$

The derivation of eigenvector centrality

Discussion: What did you notice?

- What is x(3)?

Answer: $x(3) = A(A(A\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix})))$ = $A^3x(0)$

$x(3)$ depends on the centrality of its neighbors of distance 3 or less

- What is x(t)?

Answer: $x(\mathrm{t}) = A(A \ldots (A\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix})))$ = $A^t x(0)$, $t > 0$

$x(\mathrm{t})$ depends on the centrality of its neighbors of distance t or less

Eigenvector Centrality Derivation

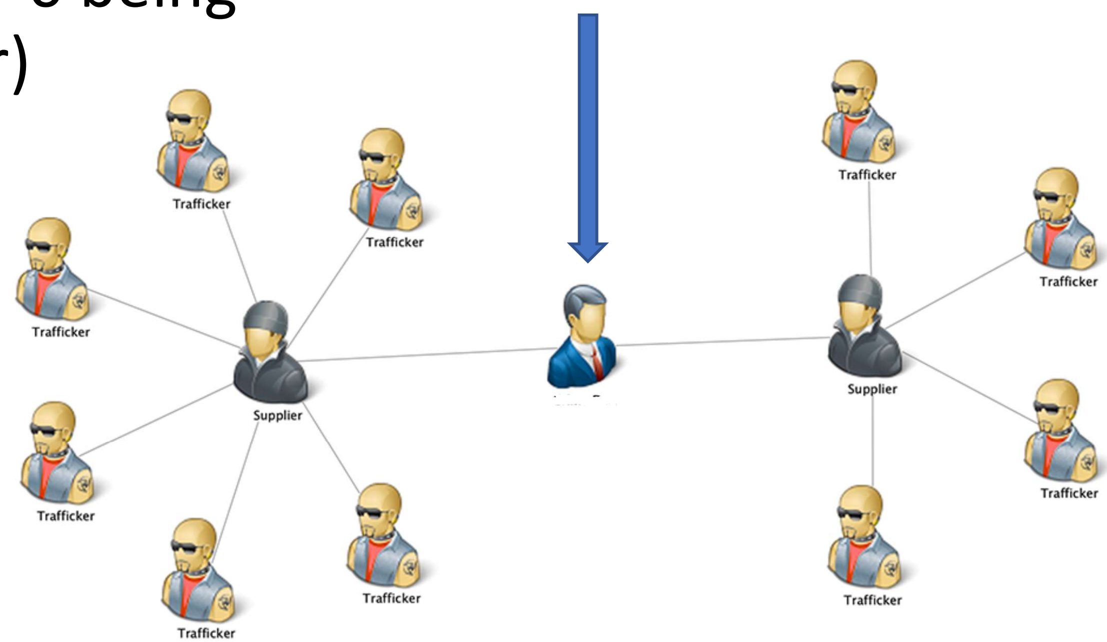
- We can consolidate the eigenvector centralities of all the nodes in a recursive formula with vectors:

$$\mathbf{x}(t) = A \cdot \mathbf{x}(t-1)$$

with the centrality at time t=0 being

$$\mathbf{x}(0) = \mathbf{1} \text{ (as a vector)}$$

- Then, we solve:
  $\mathbf{x}(t) = A^t \cdot \mathbf{x}(0)$,
  with $\mathbf{x}(0) = \mathbf{1}$



Eigenvector Centrality Derivation

Let:

- $\mathbf{x}(t) = A^t \cdot \mathbf{x}(0)$, with $\mathbf{x}(0) = 1$
- $\boldsymbol{v_k}$ are the eigenvectors of the adjacency matrix A
- $\mathbf{x}(0) = \sum_k c_k v_k$ is a linear combination of $\boldsymbol{v_k}$
- λ_1 be the largest eigenvalue.

Then

$$\mathbf{x}(t) = A^t \cdot \mathbf{x}(0) = A^t \sum_k c_k v_k = \sum_k c_k \lambda_k^t v_k =$$

$$= \lambda_1^t \sum_k c_k \frac{\lambda_k^t}{\lambda_1^t} v_k = \lambda_1^t \left(c_1 \frac{\lambda_1^t}{\lambda_1^t} v_1 + c_2 \frac{\lambda_2^t}{\lambda_1^t} v_2 + c_3 \frac{\lambda_3^t}{\lambda_1^t} v_3 \ldots\right)$$

$\mathbf{x}(t) \rightarrow \lambda_1^t c_1 v_1$

since $\frac{\lambda_k^t}{\lambda_1^t} \rightarrow 0$

as t $\rightarrow \infty$ (as you repeat the process)

Eigenvector Centrality

- Thus, the eigenvector centrality is

$$\boldsymbol{x}(t) = \lambda_1^t c_1 \boldsymbol{v}_1$$

where $\boldsymbol{v}_\mathbf{1}$ is the eigenvector corresponding to the largest eigenvalue λ_1^t

- So the eigenvector centrality (as a vector), $\boldsymbol{x}(t)$, is a multiple of the eigenvector $\boldsymbol{v}_1$, i.e. $\boldsymbol{x}(t)$ is an eigenvector of A.

$$\mathrm{A}\, \boldsymbol{x}(t) = \lambda_1^t \boldsymbol{x}(t)$$

- Meaning that the eigenvector centrality of each node is given by the entries of the leading eigenvector (the one corresponding to the largest eigenvalue $\lambda = \lambda_1^t$)

Is it well defined?

- That is:
  - Is the eigenvector guaranteed to exist?
  - Is the eigenvector unique?
  - Is the eigenvalue unique?
  - Can we have negative entries in the eigenvector?
- We say that a matrix/vector is positive if all of its entries are positive
- Perron-Frobenius theorem: A real square matrix with positive entries has a *unique largest real eigenvalue* and that the *corresponding eigenvector has strictly positive components*
- Perron-Frobenius theorem applies to positive matrices (but it gives similar information for nonnegative ones)

Perron-Frobenius theorem for nonnegative symmetric (0,1)-matrices

Let A ∈ $R^{n\ X\ n}$ be symmetric (0,1)-nonnegative, then

- there is a unique maximal eigenvalue λ_1 of the matrix A (for any other eigenvalue λ, we have $\lambda < \lambda_1$, with the possibility of $|\lambda| = \lambda_1$ for nonnegative matrices)
- λ_1 is real, simple (i.e., has multiplicity one), and positive (trace is zero so some are positive and some negative),
- the associated eigenvector is nonnegative (and there are no other nonnegative ones since all eigenvectors are orthogonal)

If you have not seen this and its proof in linear algebra, see a proof on pages 346-347 of Newman’s textbook

Note

Consider the vectors computed: $\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\3\\0\\3\\2\end{pmatrix}, \begin{pmatrix}9\\9\\8\\0\\8\\6\end{pmatrix}, \begin{pmatrix}25\\25\\24\\0\\24\\16\end{pmatrix}$

- In finding the eigenvector, these vectors get normalized as they are computed using the power method from Linear Algebra, and eventually converge to a normalized eigenvector as well.
- Note that $\frac{x_i(2)}{x_i(1)} \neq \frac{x_i(3)}{x_i(2)}$, where x_i is the i^{th} entry, however, the ratios will converge to λ_1

Conclusion: Eigenvector Centrality

- Eigenvector Centrality:
  - a generalized degree centrality (takes into consideration the global network)
  - extremely useful, one of the most common ones used for non-oriented networks
  - $C_i \propto \sum_{\mathrm{j}} C_j$ or $C_i = \lambda^{-1} \sum_j A_{ij} C_j$ or $C_i = \sum_{ij \,\epsilon\, E(G)} C_j$

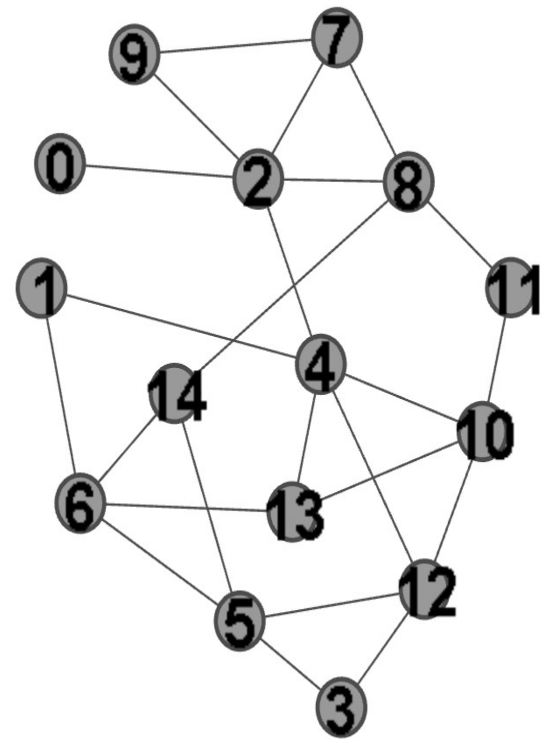
- Why is Eigenvector Centrality not commonly used for directed graphs?
  - Adjacency matrix is asymmetric…use left or right leading eigenvector?
  - Choose right leading eigenvector…importance bestowed by vertices pointing toward you (same problem with left).
    - Any vertex within degree zero has centrality value zero and "passes" that value to all vertices to which it points.

  - The fix: Katz centrality

Extra example 2 (Adjacency matrix, eigenvector centrality and the graph)

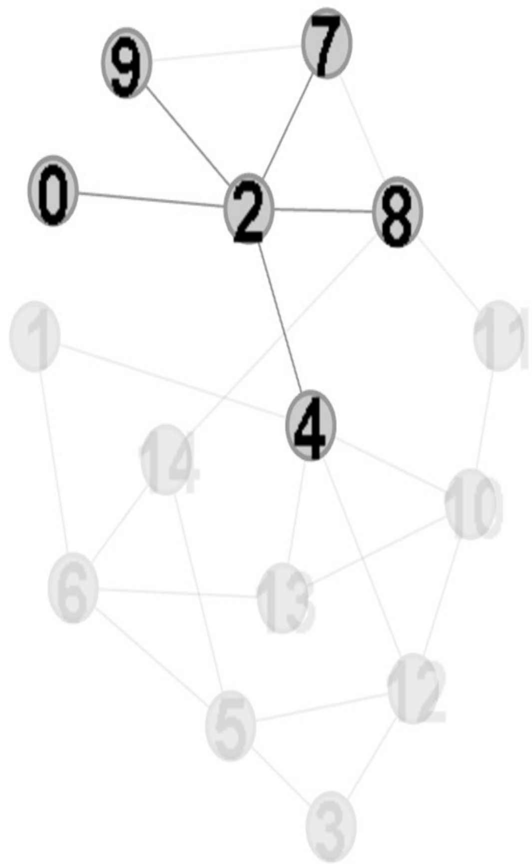
0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	1.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.
1.	0.	0.	0.	1.	0.	0.	1.	1.	1.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	1.	0.	0.
0.	1.	1.	0.	0.	0.	0.	0.	0.	0.	1.	0.	1.	1.	0.
0.	0.	0.	1.	0.	0.	1.	0.	0.	0.	0.	0.	1.	0.	1.
0.	1.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	1.	1.
0.	0.	1.	0.	0.	0.	0.	0.	1.	1.	0.	0.	0.	0.	0.
0.	0.	1.	0.	0.	0.	0.	1.	0.	0.	0.	1.	0.	0.	1.
0.	0.	1.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	1.	0.	0.	0.	0.	0.	0.	1.	1.	1.	0.
0.	0.	0.	0.	0.	0.	0.	0.	1.	0.	1.	0.	0.	0.	0.
0.	0.	0.	1.	1.	1.	0.	0.	0.	0.	1.	0.	0.	0.	0.
0.	0.	0.	0.	1.	0.	1.	0.	0.	0.	1.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.	1.	0.	1.	0.	0.	0.	0.	0.	0.

0: 0.08448651593556764,
1: 0.192860842646263 3,
2: 0.3011603786470362,
3: 0.17530527234882679,
4: 0.40835121533077895,
5: 0.2865100597893966,
6: 0.2791290343288953,
7: 0.1931920790704947,
8: 0.24881035953707603,
9: 0.13868390351302598,
10: 0.336067959653752,
11: 0.16407815738375933,
12: 0.33838887484747293,
13: 0.287139163962487 1,
14: 0.2284802392563313 5



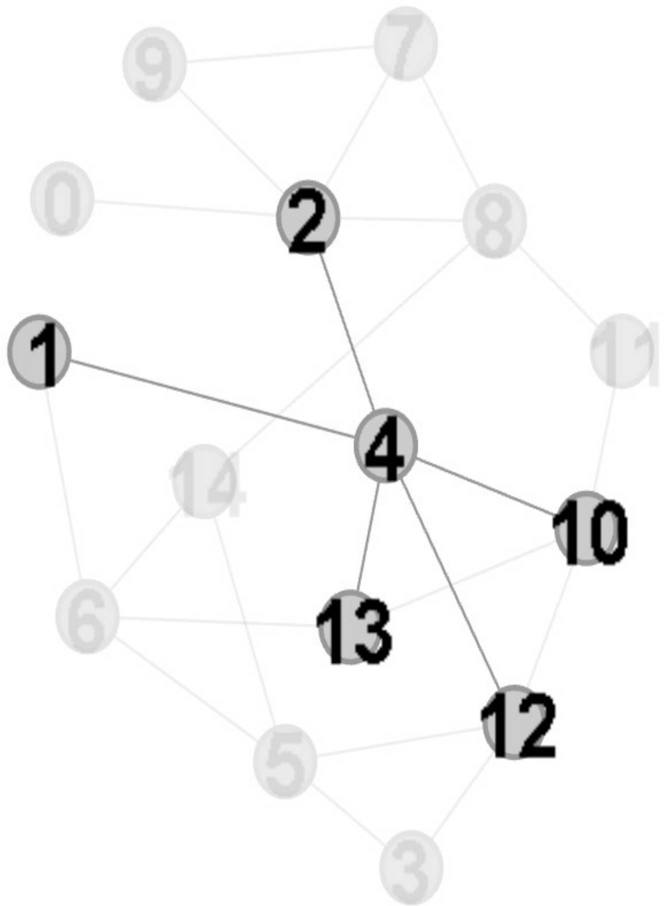
Extra example 2 (Eigenvector centrality)

C2 = 0.3011603786470362



Adjacent to vertices of small degree

C4 = 0.40835121533077895



Adjacent to vertices of large degree