Katz Centrality for directed graphs
Learning Outcomes

• Understand how Katz centrality is an extension of Eigenvector Centrality to directed graphs.
• Compute Katz centrality per node.
• Interpret the meaning of the values of Katz centrality.
## Recall: Centralities

<table>
<thead>
<tr>
<th>Quality: what makes a node important (central)</th>
<th>Mathematical Description</th>
<th>Appropriate Usage</th>
<th>Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lots of one-hop connections from $v$</td>
<td>The number of vertices that $v$ influences directly</td>
<td>Local influence matters</td>
<td>Degree $\deg(i)$</td>
</tr>
<tr>
<td>Lots of one-hop connections from $v$ relative to the size of the graph</td>
<td>The proportion of the vertices that $v$ influences directly</td>
<td>Local influence matters</td>
<td>Degree centrality $C_i = \frac{\deg(i)}{</td>
</tr>
<tr>
<td>Lots of one-hop connections to high centrality vertices</td>
<td>A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality)</td>
<td>For example, when the people you are connected to matter.</td>
<td>Eigenvector centrality (recursive formula): $C_i \propto \sum_j C_j$</td>
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What changes in directed graphs?!
Recall: Strongly connected

**Definition:** A directed graph $D = (V, E)$ is **strongly connected** if and only if, for each pair of nodes $u, v \in V$, there is a path from $u$ to $v$.

How do we compute centralities if the graph is not strongly connected?

- For example, the Web graph is not strongly connected since
  - there are pairs of nodes $u$ and $v$, there is no path from $u$ to $v \text{ and from } v \text{ to } u$.
- This presents a challenge for nodes that have an in-degree of zero
  - Why? What is the cascading effect?
  - What is a solution?

[Diagram of a directed graph showing nodes and edges, along with the Wikipedia link for Directed acyclic graph.]
Katz Centrality

• Recall that the eigenvector centrality $x(t)$ is a weighted degree obtained from the leading eigenvector of $A$: $A x(t) = \lambda_1 x(t)$, so its entries are

$$x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j$$

Thoughts on how to adapt the above formula for directed graphs (maybe not all being strongly connected)?

• Katz centrality: $x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta$, where $\beta$ is a constant initial weight given to each vertex so that vertices with zero in degree (or out degree) are included in calculations.

• After this augmentation, a random surfer on a particular webpage, has two options:
  ✓ He randomly chooses an out-link to follow ($A_{ij}$)
  ✓ He jumps to a random page ($\beta$)

Does $\beta$ have to be the same for each vertex?

• An extension: $\beta_i$ is an initial weight given to vertex $i$ as a mechanism to differentiate vertices using some quality not modeled by adjacencies. Vertices with zero in degree (or out degree) will be included in calculations.
Katz Centrality

Does
• Generalize the concept of eigenvector centrality to directed networks that are not strongly connected

Does not
• Control for the fact that a high centrality vertex imparts high centrality on those vertices “downstream,” or all those vertices reachable from that high centrality vertex → PageRank
## Updated Overview:

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<td>Lots of one-hop connections to high out-degree vertices (where each vertex has some pre-assigned weight)</td>
<td>A weighted degree centrality based on the out degree of the neighbors</td>
<td>Directed graphs that are not strongly connected</td>
<td>Katz centrality ( x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta ), Where ( \beta ) is some initial weight</td>
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