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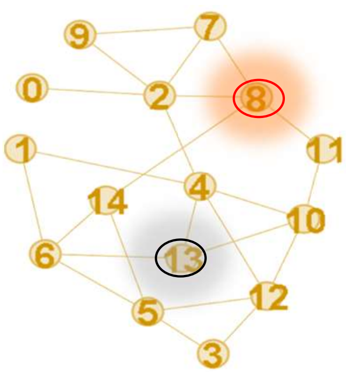
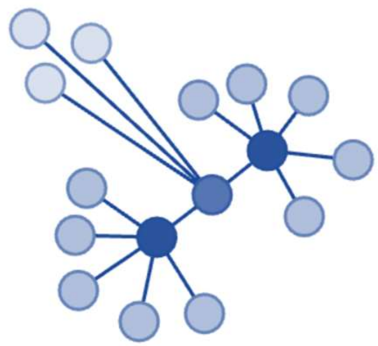
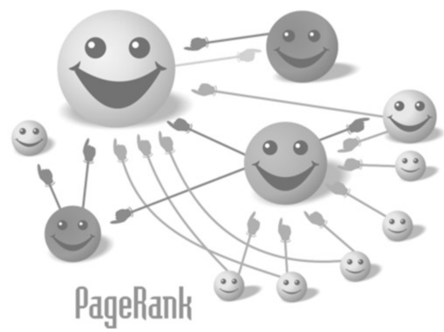
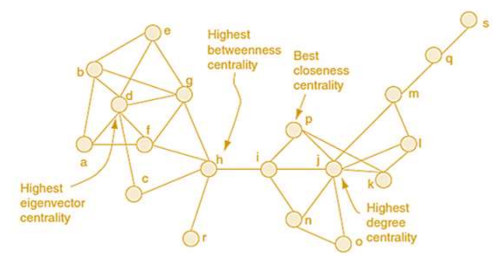








MA4404 Complex Networks

Katz Centrality for directed graphs

Learning Outcomes

- Understand how Katz centrality is an extension of Eigenvector Centrality to directed graphs.
- Compute Katz centrality per node.
- Interpret the meaning of the values of Katz centrality.

Recall: Centralities

Quality: what makes a node important (central)	Mathematical Description	Appropriate Usage	Identification
Lots of one-hop connections from v	The number of vertices that v influences directly	Local influence matters Small diameter	Degree $\deg(i)$
Lots of one-hop connections from v relative to the size of the graph	The proportion of the vertices that v influences directly	Local influence matters Small diameter	Degree centrality $\mathrm{C_i} = \frac{\deg(i)}{\lvert V(G) \rvert}$
Lots of one-hop connections to high centrality vertices	A weighted degree centrality based on the weight of the neighbors (instead of a weight of 1 as in degree centrality)	For example, when the people you are connected to matter.	Eigenvector centrality (recursive formula): $C_i \propto \sum_{\mathrm{j}} C_j$

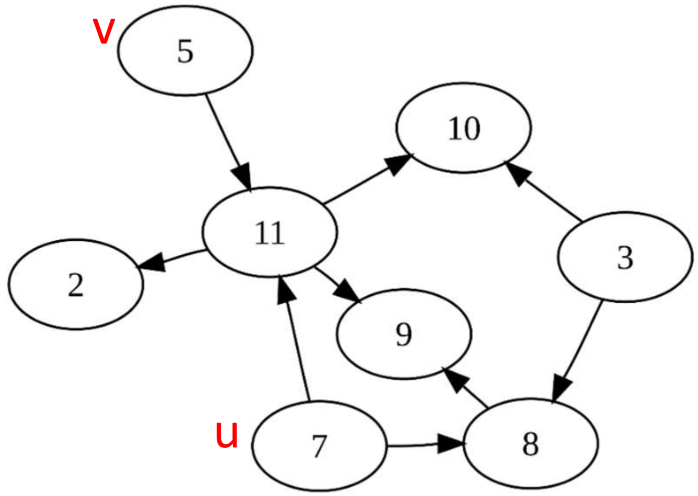
What changes in directed graphs?!

Recall: Strongly connected

Definition: A directed graph *D* = (*V*, *E*) is strongly connected if and only if, for each pair of nodes *u*, *v* ∈ *V*, there is a path from *u* to *v*.

How do we compute centralities if the graph is not strongly connected?

- For example, the Web graph is not strongly connected since
  - there are pairs of nodes *u* and *v*, there is no path from *u* to *v* *and* from *v* to *u*.
- This presents a challenge for nodes that have an in-degree of zero
  - Why? What is the cascading effect?
  - What is a solution?



http://en.wikipedia.org/wiki/Directed_acyclic_graph

Katz Centrality

- Recall that the eigenvector centrality $\mathbf{x}(t)$ is a weighted degree obtained from the leading eigenvector of A: $\mathrm{A}\,\mathbf{x}(t) = \lambda_1 \mathbf{x}(t)$, so its entries are

$$x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j$$

Thoughts on how to adapt the above formula for directed graphs (maybe not all being strongly connected)?

- Katz centrality: $x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta$, where β is a constant initial weight given to each vertex so that vertices with zero in degree (or out degree) are included in calculations.
- After this augmentation, a random surfer on a particular webpage, has two options:
  - ✓ He randomly chooses an out-link to follow (A_{ij})
  - ✓ He jumps to a random page (β)

Does β have to be the same for each vertex?

- An extension: β_i is an initial weight given to vertex i as a mechanism to differentiate vertices using some quality not modeled by adjacencies. Vertices with zero in degree (or out degree) will be included in calculations.

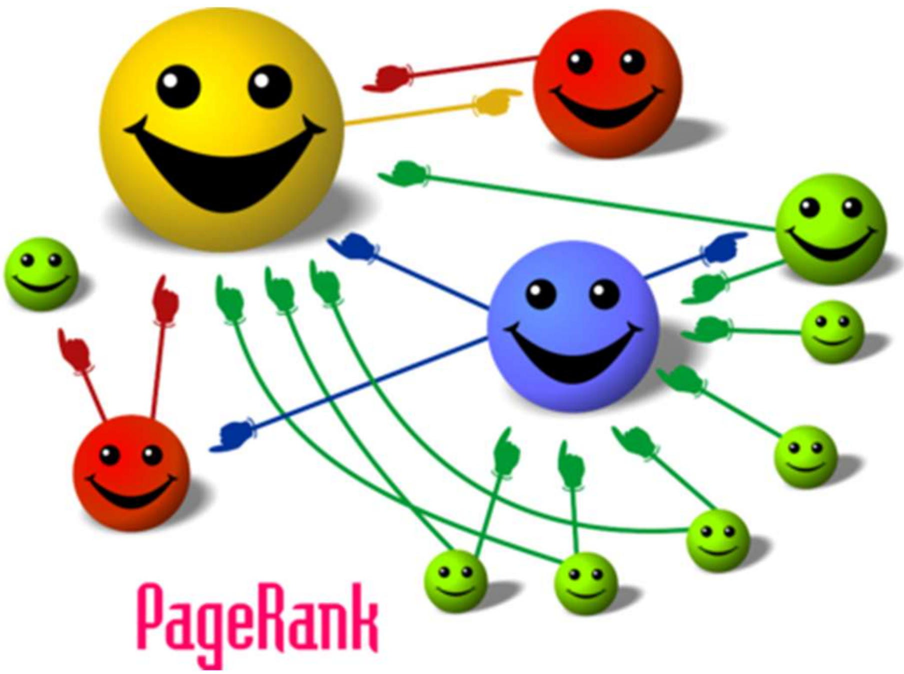
Katz Centrality

Does

- Generalize the concept of eigenvector centrality to directed networks that are not strongly connected

Does not

- Control for the fact that a high centrality vertex imparts high centrality on those vertices "downstream," or all those vertices reachable from that high centrality vertex → PageRank



Updated Overview:

Quality: what makes a node important (central)	Mathematical Description	Appropriate Usage	Identification
Lots of one-hop connections relative to the size of the graph	The proportion of the vertices that v influences directly	Local influence matters Small diameter	Normalized degree centrality $C_i = \deg(i)$
Lots of one-hop connections to high centrality vertices	A weighted degree centrality based on the weight of the neighbors	For example when the people you are connected to matter.	Eigenvector centrality $C_i \propto \sum_j C_j$
Lots of one-hop connections to high out-degree vertices (where each vertex has some pre-assigned weight)	A weighted degree centrality based on the out degree of the neighbors	Directed graphs that are not strongly connected	Katz centrality $x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j + \beta$, Where β is some initial weight