1 The Foundations: Logic and Proofs

1.4 Predicates and Quantifiers

1. predicate logic is the area of logic that deals with predicates (and quantifiers)

2. the predicate is the property that the subject of the statement can have. Example of a predicate: "is an integer"

3. the propositional function P at x is the statement that involves the variable x, that will be a proposition when x is assigned a value. For example: $P(x) : x$ is an integer.

4. propositional functions may have more than one variable, for example $P(x, y, z)$

5. a quantifier expresses the extend to which the predicate is true over a range of values, and it helps create a proposition from propositional function.
   Example: $\forall x \in \mathbb{Z}, x^2 \in \mathbb{N}$

6. the quantifiers are:

   - the universal quantifier: $\forall x$. This means that for all values of x $P(x)$ is true. Example: $\forall x \in \mathbb{N}, x^3 \geq 0$.
   - existential quantifier: $\exists x$. This means that there is at least one value of x so that $P(x)$ is true. Example: $\exists x \in \mathbb{N}, x^3 \geq 10$.
   - uniqueness quantifier: $\exists!$. This means that there is *exactly* one value of x so that $P(x)$ is true. Example: $\exists! x \in \mathbb{N}, x^3 = 0$.

7. a counterexample to $\forall x, P(x)$ is an element x for which $P(x)$ is false. For example,

   let $P(x) : \forall x \in \mathbb{N}, x^3 \geq 10$. A counterexample is the value $x = 0$, or $x = 2$.

8. note that "$\forall x \in A,\ P(x)$" is the same as saying " if $x \in A$, then $P(x)$", so it is an implication

9. note that if A is empty in "$\forall x \in A,\ P(x)$", then the predicate is true since no value of x will make $P(x)$ false.

10. when we have $\forall x \in A,\ P(x)$ or $\exists x \in A,\ P(x)$, then we assume that there is some value in the domain. What that says is that $\forall x \in A,\ P(x)$ is true if the set A is empty as well, but we generally assume that A is nonempty (however one should check if A is empty or not).

11. quantifiers have higher precedence than all logical operators from propositional calculus. Example: $\forall x P(x) \vee Q(x)$ means $\Big(\forall x P(x)\Big) \vee Q(x)$ and not $\forall x \Big(P(x) \vee Q(x)\Big)$

12. if a quantifier is used on a variable x, we say that x is bound, and it is free otherwise

13. two statements involving quantifiers and connectives are logically equivalent iff they have the same truth values for the same predicates and variables that are substituted in (any value of the domain could be used).

14. **Negating quantifiers:**

- $\neg\Big(\forall x P(x)\Big) \equiv \exists x \neg P(x)$ $\Big($or $\neg\Big(\forall x \in A,\ P(x)\Big) \equiv \exists x \in A,\ \neg P(x)\Big)$
- $\neg\Big(\exists x P(x)\Big) \equiv \forall x \neg P(x)$ $\Big($or $\neg\Big(\exists x \in A,\ P(x)\Big) \equiv \forall x \in A,\ \neg P(x)\Big)$

For example: $\neg(\forall x \geq 0, x^2 \geq 1)$ is $\exists x \geq 0, x^2 < 1$.