1 The Foundations: Logic and Proofs

1.5 Nested Quantifiers

1. in this section we combine more than one quantifier in the same mathematical statement
2. $\forall x \exists y, P(x, y)$ is the same as $\forall x, Q(x, y)$ where $Q(x, y) : \exists y, P(x, y)$
3. the order of the quantifiers is important: $\forall x \exists y, P(x, y) \not\equiv \exists y \forall x, P(x, y)$
4. Generally: if the two symbols are the same (such as $\forall x \forall y, P(x, y)$ or $\exists x \exists y, P(x, y)$) then the order of the variables doesn't matter. It is commonly written as: $\forall x \forall y, P(x, y) \equiv \forall x, y, P(x, y)$.
5. However, if the two symbols are not the same (such as $\forall x \exists y, P(x, y)$ or $\exists x \forall y, P(x, y)$) then the order matters, and the two propositions have different meanings (see table 1 page 53)
6. $\underline{\forall x \exists y, P(x, y)}$ means that no matter what x you choose, there is y that makes $P(x, y)$ true (most of the times y depends on x). Example: $\forall x \exists y, x + y = 0$. To see this, let $y = -x$.
7. now, $\underline{\exists x \forall y, P(x, y)}$ means that you could find some x, such that no matter what y you choose (this y cannot depend on x, it should be any value y) $P(x, y)$ is true. Example: $\exists x \forall y, x + y = 0$. This is not true, because you can always find some y that makes it false. However, the proposition $\exists x \forall y (y \neq 0), \frac{x}{y} = 0$ is true.
8. additive inverse of x is $-x$ (so that adding them up you get 0)
9. multiplicative inverse of x is $\frac{1}{x}$ (so that when you multiply them you get 1)
10. when negating statements involving more quantifiers, each quantifier gets negated so that if it was $\exists$ it becomes $\forall$, and backwards. Don't forget to negate the $P(x, y)$ part that follows.
11. the negation of $\vee$ is $\wedge$, and the negation of $\wedge$ is $\vee$
12. the negation of $\geq$ is $<$, and similarly for the other inequality signs
13. **the negation of $p \to q$ is NOT $\neg p \to \neg q$.** But rather: the <u>negation of $p \to q$</u> is the <u>negation of $\neg p \vee q$</u> which is logically equivalent to <u>$p \wedge \neg q$</u>