1 The Foundations: Logic and Proofs

1.6 Rules of Inference

Rules of Inference

1. an argument in propositional logic is a sequence of propositions that take the premise(s) to prove the conclusion(s) (an argument works for particular propositions)

2. an argument form in propositional logic is a sequence of propositions involving propositional variables that take the premise(s) to prove the conclusion(s) (an argument form is true no matter what particular propositions are used for the propositional variables, it is like a "rule" that works for all propositional variables used)

3. a rule of inference is a simple valid argument form that can be used as laws

4. rules of inference

   (a) law of detachment (modus ponens): $[p \wedge (p \to q)] \to q$

   (b) modus tollens: $[\neg q \wedge (p \to q)] \to \neg p$

   (c) hypothetical syllogism: $[(p \to q) \wedge (q \to r)] \to (p \to r)$

   (d) disjunctive syllogism: $[(p \vee q) \wedge \neg p] \to q$

   (e) addition: $p \to (p \vee q)$ or similarly: $q \to (p \vee q)$

   (f) simplification: $[p \wedge q] \to q$ or similarly $[p \wedge q] \to p$

   (g) conjunction: $[(p) \wedge (q)] \to (p \wedge q)$ (if we know p and also q, then we have $p \wedge q$)

   (h) resolution: $[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$

5. a fallacy of affirming the conclusion is an incorrect reasoning in proving $p \to q$ by starting with assuming q and proving p. For example: Show that if $x+y$ is odd, then either x or y is odd, but not both. A fallacy of affirming the conclusion argument would start with: "Assume that either x or y is odd, but not both. Then.."

6. a fallacy of denying the hypothesis is an incorrect reasoning in proving $p \to q$ by starting with assuming $\neg p$ and proving $\neg q$. For example: Show that if x is irrational, then $x/2$ is irrational. A fallacy of denying the hypothesis argument would start with: "Assume that x is rational. Then..."

Definition: We define x to be an even integer if $x = 2k, \exists k \in \mathbb{Z}$. Similarly, we define x to be an odd integer if $x = 2k + 1, \exists k \in \mathbb{Z}$.

Rules of Inference for Quantified statements

1. universal instantiation: knowing $\forall x, P(x)$ we can deduce $P(a)$ for any value a that we need.

2. universal generalization: knowing $P(a)$ for an arbitrary a we can deduce that $\forall x P(x)$ since a was arbitrary.

   Example: **Prove that the square of an even integer is also even.**
   Proof: Let a be an even integer (this is the universal instantiation). Then $a = 2k, \exists k \in \mathbb{Z}$. (the definition of an even integer) Then $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even (reasoning). Thus a^2 is even, and so there is the square of an even integer is also even (universal generalization).

3. existential instantiation: knowing $\exists x, P(x)$ we can deduce $P(a)$ for some value a (sometimes you don't know the value of a but we know of its existence)

4. existential generalization: knowing $P(a)$ for some value of a we can deduce that $\exists x P(x)$ since there is at least one value for which it is true, for example the value a

   Example: **Prove that there is an even integer who is the sum of two odd numbers.**
   Proof: Let 1 and 5 be the two odd numbers (existential instantiation). Note that $1 + 5 = 6$ is even (reasoning). Since the even number 6 can be written as the sum of two odd numbers, it follows that there is an even integer who is the sum of two odd numbers.