1 The Foundations: Logic and Proofs

1.7 Introduction to Proofs

1. a proof is a valid argument that establishes the truth of a theorem/proposition/lemma/corollary

2. an axiom (or postulate) is a statement that is assumed to be true without a proof

3. a conjecture is a statement that is being proposed to be true (usually based on partial results, intuition or heuristic argument) and it could become a theorem if it gets proved

4. Proof Methods in proving $P(x) \to Q(x)$ or equivalent statements

   (a) Direct Proof: *Assume $P(x)$ and prove $Q(x)$* (that's why it is the *direct* proof). The proof will generally start by choosing an arbitrary element (that is you can't assume anything particular about that element) of the domain, say a, (universal instantiation) and then proving the result about a. This far, we showed that $P(a) \to Q(a)$. Once this is done, since a was chosen arbitrary, we can then generalize it back and say that for $\forall x$ in the domain the result is T.

   (b) Contrapositive (Proof by Contraposition): *Assume $\neg Q(x)$ and prove $\neg P(x)$* (look at the truth tables to see that they are equivalent)

   (c) Vacuous Proof: In proving $P(x) \to Q(x)$ we find that *$P(x)$ is always F* for all values of x (so we get the implication: $F \to T/F$ which is always T)
   Example: If $x^2 \leq -2$, then x is even. (This is true since x^2 cannot be less than or equal to -2)

   (d) Trivial Proof: In proving $P(x) \to Q(x)$ we find that *$Q(x)$ is always T* (without even using the given facts of $P(x)$).
   Example: For positive numbers x, we have that $x^2 + 2 \geq 2$. (This is true no matter if x is positive or not)

   (e) contradiction: we prove $P(x) \to Q(x)$ by proving that *$P(x)$ and $\neg Q(x)$ imply a contradiction*. Note that this is true since we are proving that if we assume $\neg(P(x) \to Q(x))$ we get a contradiction, which makes $P(x) \to Q(x)$ a tautology. To see this, observe that $\neg(P(x) \to Q(x)) \equiv \neg(\neg P(x) \vee Q(x)) \equiv P(x) \wedge \neg Q(x))$

5. Common mistakes:

   (a) fallacy of affirming the conclusion (see section 1.5)

   (b) fallacy of denying the hypothesis (see section 1.5)

   (c) circular reasoning: a statement is proved using itself or a statement equivalent to it (see example 18 page 84)