1 The Foundations: Logic and Proofs

1.8 Proof Methods and Strategy

1. The result is of the form $\forall x(P(x) \to Q(x))$ or equivalent statements

   - Direct Proof: *assume* $P(x)$ *and prove* $Q(x)$
   - Contrapositive: *assume* $\neg Q(x)$ *and prove* $\neg P(x)$
   - Vacuous Proof: find that $P(x)$ *is always* F for all values of x
   - Trivial Proof: find that $Q(x)$ *is always* T (without using $P(x)$).
   - contradiction: prove that $P(x)$ *and* $\neg Q(x)$ *imply a contradiction*

   within any of the above methods, one might have to use the following:

   (a) proof by cases: if a single argument will not be valid for all values of x (every x of the domain needs to belong to one case). Cases should be considered if there is no obvious way to start a proof, since it may seem like not enough information is given in the hypotheses. The cases are usually given by the statement, depending on what the result says, however common cases are:

   - (1) x is even and (2) x is odd
   - (1) $x \geq 0$ and (2) $x < 0$ (sometimes $x = 0$ should be considered as Case (3))
   - (1) $x \in \mathbb{Q}$ and (2) $x \notin \mathbb{Q}$

   "WLOG" (Without loss of generality) should be used if two or more cases are similar, so that you wouldn't repeat the exact same proof.

   (b) exhaustive proof: to prove the result, one may prove every possible example (that is if the number of examples is relatively small). It is not an elegant proof technique

2. Existence Proof: The result is of the form $\exists x(P(x) \to Q(x))$. Find one example α for which both $P(\alpha)$ and $Q(\alpha)$ are true. There are two forms of existence proofs:

   (a) constructive: when the value α is actually found

   (b) nonconstructive: when the existence of such value is proved, without specifying the value

3. Uniqueness Proof: The result is of the form $\exists! x(P(x) \to Q(x))$. It has two parts:

   - existence: show that there is an element (either constructive or not)
   - uniqueness: assume that there is another element, say $y \neq x$, and prove that either you get a contradiction or that $y = x$ (which is a contradiction as well)