10 Graphs

10.1 Graphs and Graph Models

1. A graph $G = (V(G), E(G))$ consists of a set $V(G)$ of vertices, and a set $E(G)$ of edges (edges are pairs of elements of $V(G)$)

2. An edge is present, say $e = \{u, v\}$ (or simply just uv) for $\exists u, v \in V(G)$, if there is a particular pre-identified property between the two vertices:

3. Examples:

   - Friendship graph (or acquaintanceship graph): $V(G)$ is a set of people, and an edge is present if the two people are friends/know each other (These "social graphs" are associated with the phrase "six degrees of separation": the acquaintanceship graph connecting the entire human population has a diameter of six or less, i.e. any two random people are connected by a chain of at most 6 friendships, i.e. 5 intermediate vertices)
   - Collaboration graph (centered on Paul Erdos, who was the most prolific mathematician): $V(G)$ is the set of mathematicians, and an edge between two vertices is present if the two people co-wrote an article. The Erdos number of a vertex is the minimum number of edges traveled from Erdos's node to the particular node. Erdos himself has an Erdos number of 0. All those who co-authored a paper with him have Erdos number 1, and so on. The graph is not connected, as some people work alone.
   - Hollywood graph (centered at Kevin Bacon): $V(G)$ is the set of actors, and two vertices are adjacent if the corresponding actors worked together on a movie. There is a similar notion of Bacon's number. This graph is connected, and it has about half a million vertices, representing actors (most actors can reach Bacon's vertex on average in 3 steps).
   - There is now an Erdos-Bacon graph as well, with the sum of the values that it takes a person to get to Erdos and to Bacon (smallest number is 3)

4. An infinite graph is a graph with an infinite vertex set. Otherwise it is finite

5. A (simple) graph is a graph such that each edge connects two different vertices (i.e. no orientation, no parallel edges, and also no loops)

6. A multigraph allows parallel edges (multiple edges between the same two vertices)

7. A pseudograph is a graph that contains loops and possibly parallel edges

8. A digraph (or directed graph) $D = (V(D), E(D))$ consist of a set $V(D)$ of vertices, and a set $E(D)$ of oriented arcs as ordered pairs, so $(u, v) \neq (v, u)$ (also written as so $uv \neq vu$)

9. An oriented graph is obtained by applying an orientation to a simple

10. A directed multigraph is a directed graph that may have multiple arcs and/or loops

11. Digraphs examples

    - Round Robin Tournament digraphs: they represent tournaments in which each team plays each of the other teams. Then $V(D)$ is the set of teams, and an arc (x, y) means that team x beat team y. This is a (simple) digraph, since there are no loops (no team plays against itself), and there are no multiple arcs (for each pair of teams one beats the other)
    - Call digraphs: $V(D)$ is the set of phone numbers, and an arc (x, y) represents a phone call, where x called y. This is a directed multigraph since more than one call can be placed between a pair of numbers x and y. However, there are no loops.
    - Web digraph: $V(D)$ is the set of webpages, and a link between two pages is represented by an arc xy (namely that page x links to page y) This could have multiple arcs, and possibly loops.

12. A mixed graph allows directed and undirected edges,loops and multiple edges

13. Road maps: $V(D)$ is the set of intersections of roads, and the arcs between two vertices represent streets going in that direction.

14. Most of the graphs we work with are simple graphs, and so we refer to them just as graphs (versus simple graphs).

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes