10 Graphs

10.2 Graph Terminology and Special Types of Graphs

For the rest of the chapter, G denotes a graph and:

- V or $V(G)$ for its vertex set,
- E or $E(G)$ for the edge set,
- lower case letter $u, v, x, y, z, \ldots$ to denote vertices in the graph, and
- lower cases $e, f, g, \ldots$ to denote edges of the graph.

For digraphs/oriented graphs, we use D

- with $V(D)$ and $E(D)$ for vertex and edge sets, respectively
- same notation for nodes and edges as we use for graphs.

Some more terminology so we can model data and seek solutions for our questions:

1. If $uv \in E(G)$, then the vertices u and v are adjacent, and so u is incident with the edge $e = uv$, v is incident with the edge $e = uv$, and e connects u and v
2. The degree of the vertex v, $\deg v$, is the number of edges incident with v
3. If $\deg v = 0$ then v is an isolated vertex
4. If $\deg v = 1$ then v is a pendant or an end vertex
5. The Handshaking Theorem (also called The First Theorem of Graph Theory): For a simple graph $G = (V(G), E(G))$, we have that

$$\sum_{v \in V(G)} \deg v = 2|E(G)|$$

6. The theorem implies that sum of the degrees of the vertices has to be even
7. Thm: A simple graph has an even number of vertices of odd degree
8. If $vu \in D$, then v is adjacent to u, and u is adjacent from v (v is the initial vertex of the arc, and u is the terminal vertex)
9. In D, the in-degree of v, $\deg^-(v)$ is the number of arcs that point to v , and the out-degree of v, $\deg^+(v)$ is the number of arcs that point away from v
10. Thm: For a digraph $D = (V(D), E(D))$, we have that

$$\sum_{v \in V(D)} \deg^- v = \sum_{v \in V(D)} \deg^+ v = |E(D)|.$$

11. The underlying graph G of a digraph D, is the graph obtained from D by removing the orientation of the arcs (this graph could be simple or a multigraph)

12. Standard classes of **simple** graphs: they are simple graphs that follow a particular property (each class is infinite)

    - complete graph, $K_n, n \geq 1$ is the graph on n vertices that has every possible edge present
    - path, $P_n, n \geq 2$ is the graph on n vertices $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n-1$ (it has the consecutive edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n$)
    - cycle, $C_n, n \geq 3$ is the graph on n vertices $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$ where addition is performed modulo n (i.e. it has the consecutive edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1$)
    - wheel, $W_n, n \geq 3$ (or $W_{1,n}$) is the graph obtained from the cycle C_n and a vertex v, by adding an edge between v and each vertex of the cycle.
    - n-Cube $Q_n, n \geq 1$ is that graph that has vertices represent all the binary n-strings, and two edges are adjacent if the two corresponding binary strings differ in just one bit (i.e the Hamming distance between the two bits is 1).
    - complete bipartite graph, $K_{a,b}, (a, b \geq 1)$ is the graph obtained by partitioning the vertices into two subsets of cardinality a and b ($n = a + b$) and all edges between any vertex of the first partite set and the second partite set are present (so the number of edges is ab). If either a or b is 1, we call the bipartite graph a star $K_{1,a}$

13. A bipartite graph: is a graph whose vertex set is partitioned into two subsets V_1 and V_2 (say $V(G) = V_1 \cup V_2$), such that edges of the graph go between a vertex in V_1 and a vertex in V_2 (note that not every vertex of V_1 is adjacent to each vertex of V_2, unless we have a complete bipartite graph)

14. Thm: G is bipartite $\iff$ $V(G)$ can be colored with exactly two colors, where no two adjacent vertices are colored the same.

15. Thm: G is bipartite $\iff$ G contains no odd cycle.

16. A matching is a collection of edges (set of edges) so that no two edges share a common vertex. A maximum matching is a matching of maximum cardinality

17. A subgraph $H = (V(H), E(H))$ of a graph $G = (V(G), E(G))$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

18. A graph K is supergraph to the graph G if G is a subgraph of K

19. The union, $G \cup H$, of two graphs G and H is the graph whose vertex set is the set $V(G) \cup V(H)$, and the edge set is $E(G) \cup E(H)$ (not connected)

20. The intersection, $G \cap H$, of two graphs G and H is the graph whose vertex set is the set $V(G) \cap V(H)$, and the edge set is $E(G) \cap E(H)$.