10 Graphs

10.4 Connectivity

Connectivity looks at determining whether there is a a path between any two nodes. Why? So for example message can be sent between two people/computers/entities using intermediate links

1. A $u - v$ path in a graph is a sequence of edges that begin at vertex u and end at vertex v. A closed walk is a walk that starts and ends at the same vertex.

2. A $u - v$ simple path is a walk that does not repeat edges either vertices or edges

3. A circuit or a cycle is a path that begins and ends with the same vertex

4. Same terminology holds for directed graphs, where we're allowed to travel in the direction that the arc points

5. A graph is connected if there is a path between any two vertices of the graph (the k^{th} power of the adjacency matrix will give the walk of length k)

6. A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of any other connected subgraph of G (i.e. connected components are the largest connected subgraphs of G)

7. Some vertices/edges have important roles with respect to connectivity:

   - A cut vertex is a vertex in G whose removal produces more components
   - A vertex cut is a set of vertices whose removal produces more components. The cardinality of a minimum vertex cut is the vertex connectivity $\kappa(G)$
   - A bridge is an edge whose removal produces more components.
   - A edge cut is a set of edge whose removal produces more components. The cardinality of a minimum edge cut is the vertex connectivity $\lambda(G)$

8. Thm: An edge is a bridge $\iff$ it doesn't belongs to a cycle.

9. $\kappa(G) \leq \lambda(G) \leq \min deg(v)$.

10. The number of paths of length $r > 0$ from v_i to v_j is the $(i, j)^{th}$ entry of A^r .

11. Digraphs have two types of connectivity:

    (a) A digraph is strongly connected if for any two vertices of D, say u and v, there is a $u - v$ directed path and a $v - u$ directed path

    (b) A digraph is weakly connected if its underlying graph is connected

12. The strongly connected components of a digraph are the largest sub-digraphs that are strongly connected.