2 Sets, Functions, Sequences, and Sums

2.1 Sets

1. A set is a collection of objects.
   The objects of the set are called elements or members.
   Use capital letters : A, B, C, S, X, Y to denote the sets.
   Use lower case letters to denote the elements: a, b, c, x, y.
   If x is an element of the set X, we write $x \in X$.
   If x is not an element of the set X, we write $x \notin X$.

2. Describing a set

   (a) list all elements if the set consists of a small number of elements:
   $X = \{a, b, c\}$
   $S = \{1, 3, 5, \ldots\}$ – generally list the first 3 elements to give away the pattern, unless more than 3 are needed to see the pattern. ($S = \{1, 3, 5, 7, \ldots\}$ may be redundant, and $S = \{1, 3, \ldots\}$ does not enough information.)
   NOTE:
   - $A = \{1, 2, 3\} = \{2, 1, 3\} = \{1, 1, 3, 2\}$
   - $\emptyset = \{\}$ is the empty set versus $Y = \{\emptyset\} \neq \emptyset$

   (b) A set with condition(s): $S = \{x | p(x)\}$ or $\{x : p(x)\}$, that is: S contains all the elements x that satisfy the condition (or have the property) $p(x)$ – a property that depends on x.
   Ex: $A = \{x : x \text{ is even }\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.
   $S = \{x : (x - 1)(x + 2) = 0\} = \{1, -2\}$
   $T = \{x : |x| = 1\} = \{-1, 1\}$
   $X = \{x : x \text{ is a student in MA2025 }\}$.
   A more complex example: Let $A = \{1, 2, \ldots, 10\}$. Then define $B = \{x \in A : x < 7\} = \{1, 2, 3, 4, 5, 6\}$

3. Special sets
   $\mathbb{N} = \{0, 1, 2, \ldots, \}$ is the set of all positive whole numbers
   $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers (whole numbers)
   $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ is the set of the rational numbers

   $\mathbb{I}$ = the set of irrationals, for example: $\pi, \sqrt{2}, -\sqrt{3}$
   $\mathbb{R}$ = the real numbers
   $\mathbb{C}$ = the set of complex numbers: $a + bi$

4. We say that a set A is a subset of a set B if every element of A is an element of B. If A is a subset of B, we write $A \subseteq B$.

   $$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

   If a set A is not a subset of a set B, we write $A \not\subseteq B$. In this case, there is an element in the set A that is not in B.
   The empty set $\emptyset$ is a subset of every set. (vacuous proof)

5. Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$. We then write $A = B$. Note that A and B will have the same elements, but they might be expressed differently. If they are not equal then we write $A \neq B$ (and that means that either A has an element that is not in B, or that B has an element that is not in A).

6. For a set A, we say that S is a proper subset of a set B if $A \subseteq B$ and $A \neq B$, and it is denoted by $A \subset B$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

7. For a set S, the cardinality of S, $|S|$, is the number of elements in the set S. If the cardinality is a finite number, then S is said to be finite. Otherwise it is infinite. The set of natural numbers is an example of an infinite set.

8. The intervals are infinite sets, as described below. Let $a, b \in \mathbb{R}$
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b.\}$
$[a, b) = \{x \in \mathbb{R} : a \leq x < b.\}$
$(a, b] = \{x \in \mathbb{R} : a < x \leq b.\}$
$(a, b) = \{x \in \mathbb{R} : a < x < b.\}$
$(a, \infty) = \{x \in \mathbb{R} : a < x.\}$
$(-\infty, b] = \{x \in \mathbb{R} : x \leq b.\}$

9. For a set A, the power set $P(A)$ of A is the set of all subsets of A.

   Ex 1: $A = \{a, b\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \neq \{\emptyset, a, b, \{a, b\}\}$ since a, b are not sets without the curly braces. $|\mathcal{P}(A)| = 4 = 2^2 = 2^{|A|}$.

   Ex 2: $C = \{\emptyset, \{\emptyset\}\}$. Then $\mathcal{P}(B) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. $|\mathcal{P}(B)| = 2^2 = 2^{|B|}$.

10. In general, for any set T: $|\mathcal{P}(T)| = 2^{|T|}$.

11. for a set A, we can recover A from its power set since

$$A = \bigcup_{S \in \mathcal{P}\ (\mathcal{A})} S$$

12. The cartesian product of two sets A and B is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

   Note that (a, b) is an ordered pair!! That is $(a, b) \neq (b, a)$
   Example: Let $A = \{x, y\}$ and $B = \{1, 2, 3\}$. Then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$
$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

   Note that $A \times B \neq B \times A$.

13. What is $|A \times B|$? Well, $|A \times B| = |A| \times |B| = 6$ in this case.

14. If $A = \emptyset$, then $A \times B = \emptyset$ and $B \times A = \emptyset$, for any set B.

15. The truth set of a predicate P is the set of elements (in the given domain) that makes P true.