2 Sets, Functions, Sequences, and Sums

2.3 Functions

1. a function f from A to B is an assignment of a unique value of B to each value of A. (Note that this means a function is well defined if each value of A is mapped to a unique value, and also, each value of A has to be mapped to some value of B.)

2. the set A above is called the domain, and the set B is called the codomain. A subset of the codomain makes the range of f, and that subset is the set of particular values of B that get assigned to values of A.

3. Let $a \in A$ and say that $f(a) = b$, of course with $b \in B$. Then b is called the image of a, and a is called the preimage of b. Then f is said to map a to b.

4. two functions f and g are equal if they have the same domain and codomain, and $f(x) = g(x)$ for every value x of the domain

5. two functions can be added, subtracted, divided and multiply if they have the same domain (so that the new function will be defined)

6. a function is strictly increasing iff: $\forall x, y, \Big((x < y) \rightarrow (f(x) < f(y))\Big)$.

7. a function is increasing iff: $\forall x, y, \Big((x < y) \rightarrow (f(x) \leq f(y))\Big)$.

8. a function is strictly decreasing iff: $\forall x, y, \Big((x < y) \rightarrow (f(x) > f(y))\Big)$.

9. a function is decreasing iff: $\forall x, y, \Big((x < y) \rightarrow (f(x) \geq f(y))\Big)$.

10. a function is one-to-one or injective iff (that is if and only if):

$$\forall x, y, \Big((f(x) = f(y)) \rightarrow (x = y)\Big)$$

Note that the textbook has misprints: at the bottom of page 141 it needs to be "injective" rather than "injunction", and at the top of page 145 "$x \neq y$" is extra

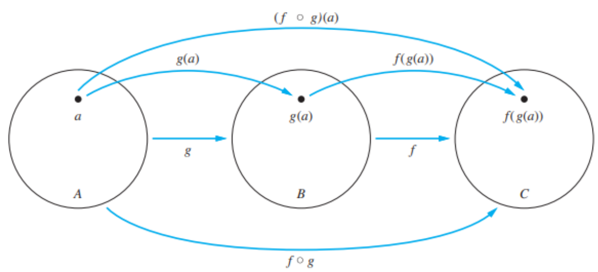
11. a function is onto or surjective iff:

$$\forall y \in B, \exists x \in A \, (f(x) = y)$$

12. a function that is both one-to-one and onto is a one-to-one correspondence or bijective. All linear functions are bijectives from reals to the reals ($f : \mathbb{R} \rightarrow \mathbb{R}$)

13. if a function $f : A \to B$ is bijective (or one-to-one correspondence) with $f(x) = y$ then there is an inverse function $f^{-1} : B \to A$ with $f(y) = x$.
    Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$, then the inverse function is $f^{-1} : \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = \frac{x-1}{2}$ (note that the expression for the inverse function is not $\frac{1}{2x+1}$)

14. the composition of two functions $g : A \to B$ and $f : B \to C$ is defined by $f \circ g : A \to C$, $(f \circ g)(x) = f(g(x))$.



15. note that $f \circ f^{-1} = f^{-1} \circ f = id$, where id is the identity function $id(x) = x$). In other words, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x values of the domain

16. the graph of the function is the set of ordered pairs $(x, f(x))$ for all x in the domain

17. the factorial function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$. For example $f(4) = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

18. the floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{R}$ is the largest integer that is less than or equal to x (Example $\lfloor 3.87 \rfloor = 3$ and $\lfloor -3.87 \rfloor = -4$

19. the ceiling function $\lceil x \rceil : \mathbb{R} \to \mathbb{R}$ is the smallest integer that is greater than or equal to x (Example $\lceil 3.27 \rceil = 4$ and $\lfloor -3.87 \rfloor = -3$

20. properties of floor and ceiling functions (n is an integer, but x is any real number):

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$