2 Sets, Functions, Sequences, and Sums

2.6 Matrices

1. A matrix $A = [a_{i,j}]$ is a rectangular array of numbers with m rows and n columns

   $(1 \le i \le m,\ 1 \le j \le n)$. Each entry of the matrix is identify by its indices, such as $a_{1,2}$ is the entry in the first row second column.

2. The sum of matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ is $A + B = [a_{i,j} + b_{i,j}]$, i.e. adding elements in the corresponding positions

3. The difference of matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ is $A - B = [a_{i,j} - b_{i,j}]$, i.e. subtracting the elements in the corresponding positions

4. The product of an $m \times n$ matrix $A = [a_{i,j}]$ and an $n \times p$ matrix $B = [b_{i,j}]$ is the $m \times p$ matrix $AB = [c_{i,j}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$:

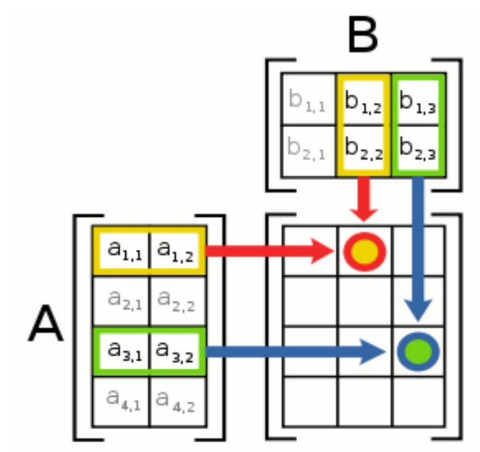

Figure 1: A diagram showing the multiplication of a 4×2 matrix A, by a 2×3 matrix B

$$\underset{2\times 3}{\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}} = \underset{2\times 4}{\begin{bmatrix} 1 & 0 & -5 & -1 \\ 3 & -2 & 11 & 3 \end{bmatrix}}$$

Figure 2: An example of multiplying a 2×3 matrix by a 3×4

5. To "divide" matrices, we introduce the inverse of a matrix, for a 2×2 matrix $A^{-1} =$

   $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$, where $ad - bc \neq 0$

6. We "divide" two matrices A and B by multiplying by the inverse: AB^{-1}

7. Note the following: $A+B=B+A$, $A-B\neq B-A$, $AB\neq BA$, , $AB^{-1}\neq B^{-1}A$

8. A transpose of a matrix $A=[a_{i,j}]$ is $A^T=[a_{j,i}]$

9. The Boolean matrices (or zero-one matrices) are matrices whose entries are 0 and 1

   - The identity matrix $I=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & 0\cdots & 1\end{bmatrix}$

   - The join of two elements: $a\vee b=\begin{cases}1 & \text{if a = 1 or b=1}\\ 0 & \text{otherwise}\end{cases}$

   - The meet of two elements: $a\wedge b=\begin{cases}1 & \text{if a =1and b=1}\\ 0 & \text{otherwise}\end{cases}$

   - The join of two $n\times m$ matrices A and B is $A\vee B=\begin{bmatrix}a_{11}\vee b_{11} & a_{12}\vee b_{12} & \cdots & a_{1m}\vee b_{1m}\\ a_{21}\vee b_{21} & a_{12}\vee b_{22} & \cdots & a_{2m}\vee b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}\vee b_{m1} & a_{m2}\vee b_{m2} & \cdots & a_{mm}\vee b_{mm}\end{bmatrix}$

   - The meet of two $n\times m$ matrices A and B is $A\wedge B=\begin{bmatrix}a_{11}\wedge b_{11} & a_{12}\wedge b_{12} & \cdots & a_{1m}\wedge b_{1m}\\ a_{21}\wedge b_{21} & a_{12}\wedge b_{22} & \cdots & a_{2m}\wedge b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}\wedge b_{m1} & a_{m2}\wedge b_{m2} & \cdots & a_{mm}\wedge b_{mm}\end{bmatrix}$

   - The Boolean product of an $m\times n$ matrix $A=[a_{i,j}]$ and an $n\times p$ matrix $B=[b_{i,j}]$ is the $m\times p$ matrix $A\odot B=[c_{i,j}]$, where $c_{ij}=(a_{i1}\wedge b_{1j})\vee(a_{i2}\wedge b_{2j})\vee\cdots\vee(a_{in}\wedge b_{nj})$

$$\mathbf{A}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}.$$

$$\begin{aligned}\mathbf{A}\odot\mathbf{B}&=\begin{bmatrix}(1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1)\\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1)\\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1)\end{bmatrix}\\ &=\begin{bmatrix}1\vee 0 & 1\vee 0 & 0\vee 0\\ 0\vee 0 & 0\vee 1 & 0\vee 1\\ 1\vee 0 & 1\vee 0 & 0\vee 0\end{bmatrix}\\ &=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}.\end{aligned}$$