2 Sets, Functions, Sequences, and Sums

2.6 Matrices

1. A matrix $A = [a_{i,j}]$ is a rectangular array of numbers with m rows and n columns

   $(1 \leq i \leq m,\ 1 \leq j \leq n)$. Each entry of the matrix is identify by its indices, such as $a_{1,2}$ is the entry in the first row second column.

2. The sum of matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ is $A + B = [a_{i,j} + b_{i,j}]$, i.e. adding elements in the corresponding positions

3. The difference of matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ is $A - B = [a_{i,j} - b_{i,j}]$, i.e. subtracting the elements in the corresponding positions

4. The product of an $m \times n$ matrix $A = [a_{i,j}]$ and an $n \times p$ matrix $B = [b_{i,j}]$ is the $m \times p$ matrix $AB = [c_{i,j}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$:

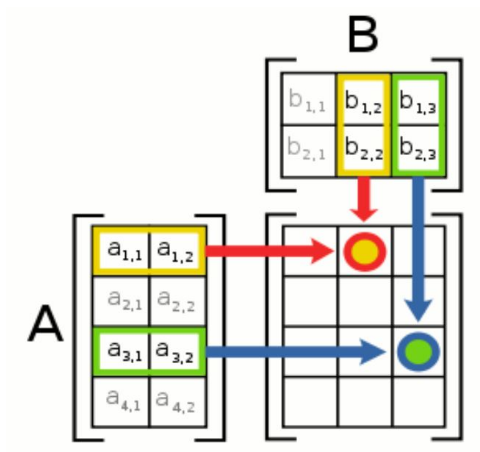

Figure 1: A diagram showing the multiplication of a 4×2 matrix A, by a 2×3 matrix B

$$\underset{2\times 3}{\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}} = \underset{2\times 4}{\begin{bmatrix} 1 & 0 & -5 & -1 \\ 3 & -2 & 11 & 3 \end{bmatrix}}$$

Figure 2: An example of multiplying a 2×3 matrix by a 3×4

5. To "divide" matrices, we introduce the inverse of a matrix, for a 2×2 matrix $A^{-1} =$

   $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$, where $ad - bc \neq 0$

6. We "divide" two matrices A and B by multiplying by the inverse: AB^{-1}

7. Note the following: $A+B=B+A$, $A-B \neq B-A$, $AB \neq BA$, , $AB^{-1} \neq B^{-1}A$

8. A transpose of a matrix $A=[a_{i,j}]$ is $A^T=[a_{j,i}]$

9. The Boolean matrices (or zero-one matrices) are matrices whose entries are 0 and 1

- The identity matrix $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0\cdots & 1 \end{bmatrix}$

- The join of two elements: $a \vee b = \begin{cases} 1 & \text{if a = 1 or b=1} \\ 0 & \text{otherwise} \end{cases}$

- The meet of two elements: $a \wedge b = \begin{cases} 1 & \text{if a =1and b=1} \\ 0 & \text{otherwise} \end{cases}$

- The join of two $n \times m$ matrices A and B is $A \vee B = \begin{bmatrix} a_{11} \vee b_{11} & a_{12} \vee b_{12} & \cdots & a_{1m} \vee b_{1m} \\ a_{21} \vee b_{21} & a_{12} \vee b_{22} & \cdots & a_{2m} \vee b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \vee b_{m1} & a_{m2} \vee b_{m2} & \cdots & a_{mm} \vee b_{mm} \end{bmatrix}$

- The meet of two $n \times m$ matrices A and B is $A \wedge B = \begin{bmatrix} a_{11} \wedge b_{11} & a_{12} \wedge b_{12} & \cdots & a_{1m} \wedge b_{1m} \\ a_{21} \wedge b_{21} & a_{12} \wedge b_{22} & \cdots & a_{2m} \wedge b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \wedge b_{m1} & a_{m2} \wedge b_{m2} & \cdots & a_{mm} \wedge b_{mm} \end{bmatrix}$

- The Boolean product of an $m \times n$ matrix $A=[a_{i,j}]$ and an $n \times p$ matrix $B=[b_{i,j}]$ is the $m \times p$ matrix $A \odot B = [c_{i,j}]$, where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{in} \wedge b_{nj})$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$