5 Induction and Recursion

5.3 Recursive Definitions & Structural Induction

In this section we introduce recursive definitions (or inductive definitions).
They are used when it is difficult/impossible to define the function directly:

1. Recursively Defined Functions use two steps to define it:

   (a) **Basis Step**: provide $f(0)$

   (b) **Inductive Step**: define $f(n)$ based on the values of f evaluated for smaller positive n values

   Example: $f(0) = 1$ and $f(n+1) = 2f(n)$, $n \geq 0$

2. A sequence is an ordered list of elements: $a_1, a_2, a_3, \ldots, a_n, \ldots$
   Example of a sequence, $s_1 : 1, 2, 3, 4, \ldots$ and a different sequence $s_2 : 1, 3, 2, 4, 5, 6, \ldots$

3. A sequence is also defined as a function $f : \mathbb{N} \rightarrow$ set of sequences, by $f(n) = a_n, n \geq 0$.
   Example for the first sequence s_1 above: $f(0) = a_0 = 1, f(1) = a_1 = 2$, and in general for s_1: $f(n) = a_n = n + 1$

4. Recursively Defined Sequences follow the recursively defined functions:

   (a) **Basis Step**: provide a_0

   (b) **Inductive Step**: define a_n based on the value of previous terms of the sequence.

   Example 1: $a_0 = 1$ and $a_{n+1} = 2a_n$, $n \geq 0$
   Example 2: $a_0 = 1$ and $a_1 = 3$ and $a_{n+1} = a_n + a_{n-1}$, $n \geq 1$

5. Fibonacci numbers: $f_0 = 1, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ (either as the Fibonacci sequence as given above, or the Fibonacci function as $f(n)$ function)

6. Recursively Defined Sets follow the recursively defined functions:

   (a) **Basis Step**: provide initial element(s) in S

   (b) **Inductive Step**: define new elements as a combination of existing elements

   Example: $1 \in S$ and if $x \in S \wedge y \in S$ then $x + y \in S$

7. Structural induction is math or strong induction applied to recursively defined functions/sets...