6 Counting

6.3 Permutations and Combinations

1. A permutation is an ordering, or arrangement, of the elements in a finite set:
   Definition: A permutation π of $A = \{a_1, a_2, \ldots, a_n\}$ is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n}$ of the elements of A (no repeats in the list).
   Example: a permutation of $A = \{1, 2, 3\}$ is $1, 3, 2$ and another one is $1, 2, 3$

   - There are $n!$ permutations of an n-element set (an n-element set is also called an n-set). For integers $n \geq 0$, the factorial $f(n) = n!$ can be recursively defined by

   $$n! = (n-1)! \cdot n \quad \textit{with} \quad 0! = 1$$

2. An r-permutation of an n-set A ($r \leq n$) is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_r}$ of **some** r-subset of A.
   Example: some 3-permutation of $A = \{1, 2, 3, 4\}$ are: $1, 4, 3$, $2, 4, 3$, $2, 3, 4$

   - There are $P(n, r)$ of r-permutation of an n-set:

   $$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)} = \frac{n!}{(n-r)!}.$$

3. An r-combination of an n-set A ($r \leq n$) is an r-subset $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ of the n-set A.
   Example: For the 4-set $A = \{1, 2, 3, 4\}$, some 3-combinations are: $\{2, 4, 3\}, \{2, 4, 5\}, \{1, 4, 3\}$

   - There are $C(n, r)$ r-combination of an n-set. We construct an r-permutation of an n-set in two steps: first take an r-combination, then take a permutation of the r-combination. It follows by the Product Rule that

   $$P(n,r) = r! \cdot C(n,r),$$

   but then

   $$C(n,r) = \frac{1}{r!} \cdot P(n,r) = \frac{n!}{r!(n-r)!}.$$

   $$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

   - Note the symmetry in combinations:

   $$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C(n, n-r)$$

   - $C(n, r)$ is commonly written $\binom{n}{r}$, which is called a *binomial coefficient*.