9 Relations

9.1 Relations and Their Properties

1. For sets A and B we define a relation from A to B to be a subset of $A \times B$
   Example: the graph of a function is a relation from the domain (A) to its range (B). Note that every function is a relation because it relates an element of A to an element of B. However not every relation a function because a relation may relate an element of A to more than just one element of B (or also because the empty relation is not a function).

2. A relation on a set A is a relation from A to A (i.e. subset of $A \times A$)

3. For a finite set A (with $|A| = n$), there are $2^{|A \times A|} = 2^{(n^2)}$ possible relations on A, namely the elements of the power set of $A \times A$

4. Properties of a relations:
   - a relation R defined on A is reflexive if $\forall a \in A$, then $(a,a) \in R, \forall a \in A$
   - a relation R defined on A is symmetric, if $(a,b) \in R$, then $(b,a) \in R$, $\forall a,b \in A$
   - a relation R defined on A is antisymmetric if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$, $\forall a,b \in A$
   - a relation R defined on A is transitive R defined on A if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, $\forall a,b,c \in A$
   - a relation R defined on A is an equivalence relation if it is reflexive, symmetric and transitive

5. Operations on relation are operations on sets:
   - union $R \cup S$
   - intersection $R \cap S$
   - difference $R - S$
   - symmetric difference $(R \oplus S = (R - S) \cup (S - R)))$
   - composition of relations: for elements $a, b, c \in A$, and two relations R and S, if $(a,b) \in R$ and $(b,c) \in S$ then $(a,c) \in S \circ R$
   - composing a relation to itself: $R^n = R^{n-1} \circ R$ with the initial condition $R^1 = R$

6. Observation: a relation R is transitive iff $R^n \subseteq R$