Chapter 1
Introduction

-- Formally, a **graph** G consists of a finite nonempty set V of objects called **vertices** (the singular is **vertex)** and a set E of 2-element subsets of V called **edges**

-- The sets V and E are the **vertex set** and **edge set** of G, respectively

-- Vertices are sometimes called **points** or **nodes** and edges are sometimes called **lines**

-- There are some who use the term **simple graph** for what we call a graph

-- Two graphs G and H are **equal** if $V(G) = V(H)$ and $E(G) = E(H)$, in which case we write $G = H$

-- If uv is an edge of G, then u and v are said to be **adjacent** in G

-- The number of vertices in G is often called the **order** of G, while the number of edges is its **size**

-- A graph with exactly one vertex is called a **trivial graph**, implying that the order of a **nontrivial graph** is at least 2

-- A graph G such as the one of Figure 1.3 (a) that has labels on vertices/edges is a **labeled graph** and Figure 1.3 (b) represents an **unlabeled graph**

-- If $e = uv$ is an edge of G, then the adjacent vertices u and v are said to be **joined** by the edge e

-- The vertices u and v are referred to as **neighbors** of each other

-- In this case, the vertex u and the edge e (as well as v and e) are said to be **incident** with each other

-- Distinct edges incident with a common vertex are **adjacent edges**

-- A graph H is called a **subgraph** of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

-- If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G

-- If a subgraph of a graph G has the same vertex set as G, then it is a **spanning subgraph** of G

-- A subgraph F of a graph G is called an **induced subgraph** of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well

-- If S is a nonempty set of vertices of a graph G, then the **subgraph of G induced by** S is the induced subgraph with vertex set S. This induced subgraph is denoted by $G[S]$. For a nonempty set X of edges, the **subgraph $G[X]$ induced by** X has edge set X and consists of all vertices that are incident with at least one edge in X. This subgraph is called an **edge-induced subgraph** of G

-- A u – v **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as where k ≥ 0 and v i and v i + 1 are adjacent for i = 0, 1, 2, …, k – 1. If u = v, then the walk W is **closed** ; while if u ≠ v, then W is **open**
-- The number of edges encountered in a walk (including multiple occurrences of an edge)
is called the **length** of the walk
-- A walk of length 0 is a **trivial walk**
-- Borrowing terminology from the Old West, we define a u – v **trail** in a graph G to be a u – v walk in which no edge is traversed more than once
-- A u – v walk in a graph in which no vertices are repeated is a u – v **path**
-- A **circuit** in a graph G is a closed trail of length 3 or more
-- A circuit that repeats no vertex, except for the first and last, is a **cycle**
-- A **k-cycle** is a cycle of length k
-- A 3-cycle is also referred to as a **triangle**
-- A cycle of odd length is called an **odd cycle**; while, not surprisingly, a cycle of even length is called an **even cycle**
-- The vertices and edges of a trail, path, circuit or cycle in a graph G form a subgraph of G, also called a **trail, path, circuit** or **cycle**
-- If G contains a u – v path, then u and v are said to be **connected** and u **is connected to** v (and v is connected to u)
-- A graph G is **connected** if every two vertices of G are connected, that is, if G contains a u – v path for every pair u, v of vertices of G
-- A graph G that is not connected is called **disconnected** (i.e. there are two verices x and y, and no x-y path)
-- A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a **component** of G
-- In this case, G is the **union** of the graphs G 1, G 2, …, G k
-- The **distance** between u and v is the smallest length of any u – v path in G and is denot
ed by d G (u, v) or simply d (u, v) if the graph G under consideration is clear
-- A u – v path of length d (u, v) is called a u – v **geodesic**
-- If the vertices of a graph G of order n can be labeled (or relabeled) v 1, v 2, …, v n so that its edges are v 1 v 2, v 2 v 3, …, v n – 1 v n, then G is called a **path** ; while if the vertices of a graph G of order n ≥ 3 can be labeled (or relabeled) v 1, v 2, …, v n so that its edges are v 1 v 2, v 2 v 3, …, v n – 1 v n and v 1 v n, then G is
called a **cycle**
-- A graph G is **complete** if every two distinct vertices of G are adjacent

-- The **complement** of a graph G is that graph whose vertex set is also V (G) and such that for each pair u, v of distinct vertices of G, uv is an edge of if and only if uv is not an edge of G
-- The graph then has n vertices and no edges ; it is called the **empty graph** of order n
-- A graph G is a **bipartite graph** if V (G) can be partitioned into two subsets U and W, called **partite sets,** such that every edge of G joins a vertex of U and a vertex of W. If this does happen, however, then we call G a **complete bipartite graph**
-- If either s = 1 or t = 1, then K s, t is a **star**
-- A graph G is a **k-partite graph** if V (G) can be partitioned into k subsets V 1, V 2, …, V k (once again called **partite sets)** such that if uv is an edge of G, then u and v belong to different partite sets. If, in addition, every two vertices in different partite sets are joined by an edge, then G is a **complete k-partite graph**
-- The complete k-partite graphs are also referred to as **complete multipartite graphs**
-- The **join** G + H consists of G ∪ H and all edges joining a vertex of G and a vertex of H
-- For two graphs G and H, the **Cartesian product** G × H has vertex set V (G × H) = V (G) × V (H), that is, every vertex of G × H is an ordered pair (u, v), where u ∈ V (G) and v ∈ V (H)
-- The graph C 4 × K 2 is often denoted by Q 3 and is called the **3-cube**
-- The graphs Q n are then called **n-cubes** or **hypercubes**
-- The n-cube can also be defined as that graph whose vertex set is the set of ordered n-tuples of 0s and 1s (commonly called **n-bit strings)** and where two vertices are adjacent if their ordered n-tuples differ in exactly Figure 1.32 : The Cartesian product of two graphs one position (coordinate
-- A **multigraph** M consists of a finite nonempty set V of vertices and a set E of edges, where every two vertices of M are joined by a finite number of edges (possibly zero
-- If two or more edges join the same pair of (distinct) vertices, then these edges are called **parallel** edges
-- In a **pseudograph**, not only are parallel edges permitted but an edge is also permitted to join a vertex to itself
-- Such an edge is called a **loop**
-- A **digraph** (or **directed graph)** D is a finite nonempty set V of objects called **vertices** together with a set E of ordered pairs of distinct vertices
-- The elements of E are called **directed edges** or **arcs**

-- Then u is said to be **adjacent to** v and v is **adjacent from** u
-- If, in the definition of digraph, for each pair u, v of distinct vertices, at most one of (u, v) and (v, u) is a directed edge, then the resulting digraph is an **oriented graph**
-- The digraph D is also called an **orientation** of G

Chapter 2
Degrees

-- The **degree of a vertex** v in a graph G is the number of edges incident with v and is denoted by deg G v or simply by deg v if the graph G is clear from the context
-- The set N (v) of neighbors of a vertex v is called the **neighborhood** of v
-- Thus deg v = | N (v)|. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** (or a **leaf**
-- The **minimum degree** of G is the minimum degree among the vertices of G and is denoted by δ(G); the **maximum degree** of G is denoted by Δ (G)
-- Since every edge of G joins a vertex of U and a vertex of W, it follows that adding the degrees of the vertices in U (or in W) gives the number of edges in G, that is, A vertex of even degree is called an **even vertex,** while a vertex of odd degree is an **odd vertex**
-- Finding the **sharpness** of a bound: find an infinite class (such as the paths on n vertices, or an infinite class that can be described by adding conditions) that achieves the equality of the bound
-- For a vertex v in a multigraph or pseudograph G, the **degree** deg v of v in G is the number of edges of G incident with v, where there is a contribution of 2 for each loop at v in a pseudograph
-- For the pseudograph G of Figure 2.4, Figure 2.4 : Illustrating degrees in a multigraph and a digraph For a vertex v in a digraph D, the **outdegree** od v of v is the number of vertices of D to which v is adjacent, while the **indegree** id v of v is the number of vertices of D from which v is adjacent
-- If δ (G) = Δ (G), then the vertices of G have the same degree and G is called **regular.** If deg v = r for every vertex v of G, where $0 \le r \le n - 1$, then G is **r-regular** or **regular of degree** r
-- A 3-regular graph is also referred to as a **cubic graph**
-- The graphs K 4, K 3, 3 and Q 3 are cubic graphs ; however, the best known cubic graph may very well be the **Petersen graph,** shown in Figure 2.6
-- The graphs H r, n described above are called **Harary graphs,** named for Frank Harary

-- If the degrees of the vertices of a graph G are listed in a sequence s, then s is called a **degree sequence** of G
-- A finite sequence of nonnegative integers is called **graphical** if it is a degree sequence of some graph
-- The **adjacency matrix** of G is the n × n matrix A = [a i j], where while the **incidence matrix** of G is the n × m matrix B = [b i j], where These matrices are shown for the graph G of Figure 2.17
-- Two u – v walks are considered **equal** if, as sequences, they are identical, term by term
-- A graph G of order at least 2 is **irregular** if every two vertices of G have distinct degrees
-- Recall that we defined a nontrivial graph G to be **irregular** if every two vertices of G have distinct degrees
-- For a graph G and a vertex v of G, define the **F-degree** F deg v of v in G as the number of copies (unlabeled subgraphs, induced or not, having the same structure) of F in G that contain v
-- A graph G is **F-regular** if every two vertices of G have the same F-degree, while G is **F-irregular** if every two vertices of G have distinct F-degrees
-- If M is a multigraph and all parallel edges joining pairs of vertices of M are replaced by a single edge, then the resulting graph G is called the **underlying graph** of M
-- Formally, two (labeled) graphs G and H are **isomorphic** (have the same structure) if there exists a one-to-one correspondence f from V (G) to V (H) such that uv ∈ E (G) if and only if f(u) f(v) ∈ E (H). In this case, f is called an **isomorphism** from G to H. Thus, if G and H are isomorphic graphs, then we say that G **is isomorphic** to H and we write G ≅ H
-- If two graphs G and H are not isomorphic, then they are called **non-isomorphic graphs**
-- A graph G is **self-complementary** if G ≅ complement(G)
-- As expected, two digraphs D 1 and D 2 are **isomorphic** if there exists a one-to-one correspondence : V (D 1) → V (D 2) such that (u 1, v 1) ∈ E (D 1) if and only if (f(u 1), f(v 1)) ∈ E (D 2)
-- One of the major consequences of knowing that isomorphism is an equivalence relation on a set of graphs is that this produces a partition of this set into equivalence classes (subsets) which are **isomorphism classes**
-- For unlabeled graphs H and G, we say that H is **isomorphic to a subgraph** of G if for any labeling of the vertices of H and G, the labeled graph H is isomorphic to a subgraph of the labeled graph G

-- There are two other isomorphisms from the graph H to itself, namely 3 and 4, defined by and An isomorphism from a graph G to itself is called an **automorphism** of G
-- This group is denoted by Aut (G) and is called the **automorphism group** o f G
-- Consequently, each of the elements of Aut (F) can be expressed in terms of and, namely, Because of this property, and a r e **generators** for the group Aut (F)
-- F and the group table for Aut (F) For a vertex v of a graph G, the set of all vertices into which v can be mapped by some automorphism of G is an **orbit** of G
-- Two vertices u and v are **similar** if they belong to the same orbit
-- On the other hand, if a graph G contains a single orbit, then every two vertices of G are similar and G is called **vertextransitive**
-- An edge e = uv of a connected graph G is called a **bridge** of G if G – e is disconnected
-- An edge e is a **bridge** of a disconnected graph if e is a bridge of some component of G
-- A **tree** is an acyclic connected graph
-- A tree containing exactly two vertices that are not end-vertices (which are necessarily adjacent) is called a **double star**
-- Another common class of trees consists of the " caterpillars ." A **caterpillar** is a tree of order 3 or more, the removal of whose endvertices produces a path called the **spine** of the caterpillar
-- There are occasions when it is convenient to select a vertex of a tree T under discussion and designate this vertex as the **root** of T
-- The tree T then becomes a **rooted tree**
-- Acyclic graphs are also referred to as **forests**
-- A spanning subgraph H of a connected graph G such that H is a tree is called a **spanning tree** of G
-- Let G be a connected graph each of whose edges is assigned a number (called the **cost** or **weight** of the edge
-- For each subgraph H of G, the **weight** w (H) of H is defined as the sum of the weights of its edges, that is, We seek a spanning tree of G whose weight is minimum among all spanning trees of G
-- Such a spanning tree is called a **minimum spanning tree**
-- The problem of finding a minimum spanning tree in a connected weighted graph is called the **Minimum Spanning Tree Problem**
-- **Kruskal' s Algorithm :** For a connected weighted graph G, a spanning tree T of G is constructed as follows : For the first edge e 1 o f T, we select any edge of G

of minimum weight and for the second edge e 2 of T, we select any remaining edge of G of minimum weight

-- **Prim' s Algorithm :** For a connected weighted graph G, a spanning tree T of G is constructed as follows : For an arbitrary vertex u for G, an edge of minimum weight incident with u is selected as the first edge e 1 of T

-- The following formula was established in 1889 by Arthur Cayley and is often referred to as the Cayley **Tree Formula**

-- By a **cofactor** of an n × n matrix M = [m ij], we mean (–1) i + j det (M ij), where det (M ij) indicates the determinant of the (n – 1) × (n – 1) submatrix M ij of M, obtained by deleting row i and column j of M

Chapter 5
Connectivity

-- A vertex v in a connected graph G is a **cutvertex** of G if G – v is disconnected

-- A nontrivial connected graph with no cut-vertices is called a **nonseparable graph**

-- A maximal nonseparable subgraph of a graph G is called a **block** of G

-- By a **vertex-cut** in a graph G, we mean a set U of vertices of G such that G – U is disconnected

-- A vertex-cut of minimum cardinality in G is called a **minimum vertex-cut**

-- For a graph G that is not complete, the **vertex-connectivity** (or simply the **connectivity**) κ(G) of G is defined as the cardinality of a minimum vertex-cut of G; (The symbol is the Greek letter kappa .)

-- if G = K n for some positive integer n, then κ (G) is defined to be n – 1

-- In general then, the **connectivity** κ (G) of a graph G is the minimum value of | U | among all subsets U of V (G) such that G – U is either disconnected or trivial

-- For a nonnegative integer k, a graph G is said to be **k-connected** if (G) ≥ k

-- An **edge-cut** in a nontrivial graph G is a set X of edges of G such that G – X is disconnected

-- An edgecut X of a connected graph G is **minimal** if no proper subset of X is an edge-cutof G

-- An edge-cut of minimum cardinality is called a **minimum edge-cut**

-- The **edge-connectivity** λ (G) of a nontrivial graph G is the cardinality of a minimum edge-cut of G, while we define λ (K 1) = 0

-- For a nonnegative integer k, a graph G is **k-edge-connected** if λ (G) ≥ k

-- These graphs are referred to as the **Harary graphs,** named for Frank Harary

-- For an integer k with 1 ≤ k ≤ d, the **kth power** G k of G is the graph with $V(G^k)=V$ (G) such that uv is an edge of G k if 1 ≤ d G (u, v) ≤ k
-- A set S of vertices of a graph G is said to **separate** two vertices u and v of G if G – S
is disconnected and u and v belong to different components of G – S
-- Such a set S is called a u – v **separating set**
-- A u – v separating set of minimum cardinality is called a **minimum** u – v **separating set**
-- An **internal vertex** of a u – v path P is a vertex of P different from u and v
-- A collection { P 1, P 2, …, P k } of u – v paths is called **internally disjoint** if every two of these paths have no vertices in common other than u and v
-- More generally, for k + 1 distinct vertices u, v 1, v 2, …, v k, a collection { P 1, P 2, …, P k } of k paths, where P i is a u – v i path (1 ≤ i ≤ k), are **internally disjoint** if every two distinct paths in the collection have only u in common
-- For 2k distinct vertices u 1, u 2, …, u k, v 1, v 2, …, v k, a collection { P 1, P 2, …, P k } of k paths, where P i is a u i – v i path (1 ≤ i ≤ k), are **disjoint** if no two distinct paths in the collection have a vertex in common
-- Such a graph G k is then called a **distance-labeled graph**
-- A path P in G k is called **proper** if every two adjacent edges in P have different labels
-- By a **proper edge labeling** of G we mean a labeling of the edges of G from the set { 1, 2, …, k } for some positive integer k such that no two adjacent edges are labeled the same

Chapter 6
Traversability

-- A circuit C in a graph G is called an **Eulerian circuit** (pronounced oy-LEER-e-an) if C contains every edge of G
-- A connected graph that contains an Eulerian circuit is called an **Eulerian graph**
-- For a connected graph G, we refer to an open trail that contains every edge of G as an **Eulerian trail**
-- This became known as the **Königsberg Bridge Problem**
-- Figure 6.7 A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle** of G
-- A **Hamiltonian graph** is a graph that contains a Hamiltonian cycle
-- A path in a graph G that contains every vertex of G is called a **Hamiltonian path** in G
-- In 1857, Hamilton invented a game called the **Icosian Game**

-- The **closure** $C(G)$ of a graph G of order n is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting graph at each stage) until no such pair remains
-- A **Hamiltonian walk** in a connected graph G is a closed spanning walk of minimum length in G

Chapter 7
Digraphs

-- Recall that a **digraph** D consists of a finite nonempty set V of objects called **vertices** and a set E of ordered pairs of distinct vertices
-- Each element of E is an **arc** or a **directed edge**
-- If a digraph D has the property that for each pair u, v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D, then D is an **oriented graph**
-- The digraph D is then referred to as an **orientation** of G
-- A digraph H is called a **subdigraph** of a digraph D if $V(H)$ is a subset of $V(D)$ and $E(H)$ a subset of $E(D)$
-- A digraph D is **symmetric** if whenever (u, v) is an arc of D, then (v, u) is an arc of D as well
-- Also, recall that if (u, v) is an arc of a digraph, then u is said to be **adjacent to** v and v is **adjacent from** u
-- The vertices u and v are also said to be **incident with** the arc (u, v)
-- The number of vertices to which a vertex v is adjacent is the **outdegree** of v and is denoted by $od(v)$
-- The number of vertices from which v is adjacent is the **indegree** of v and is denoted by $id(v)$
-- A sequence of vertices of D such that u_i is adjacent to u_{i+1} for all i $(0 \le i \le k-1)$ is called a **(directed) $u - v$ walk** in D
-- The number of occurrences of arcs on a walk is the **length** of the walk
-- A walk in which no arc is repeated is a **(directed) trail**, while a walk in which no vertex is repeated is a **(directed) path**
-- A $u - v$ walk is **closed** if $u = v$ and is **open** if $u \ne v$
-- A closed trail of length at least 2 is a **(directed) circuit**; a closed walk of length at least 2 in which no vertex is repeated except for the initial and terminal vertices is a **(directed) cycle**

-- The **underlying graph** of a digraph D is obtained by removing all directions from the arcs of D and replacing any resulting pair of parallel edges by a single edge
-- A digraph D is **connected** (sometimes called **weakly connected**) if the underlying graph of D is connected
-- A digraph D is **strong** (or **strongly connected**) if D contains both a u – v path and a v
– u path for every pair u, v of distinct vertices of D
-- The **directed distance** or, more simply, the **distance** (u, v) **from** u to v is the length of a shortest u – v path in D
-- A u – v path of length (u, v) is a u – v **geodesic**
-- An **Eulerian circuit** in a (strong) digraph D is a circuit containing every arc of D
-- An **Eulerian digraph** is a digraph containing an Eulerian circuit
-- A **tournament** is an orientation of a complete graph
-- Therefore, a **tournament** can be defined as a digraph such that for every pair u, v of distinct vertices, exactly one of (u, v) and (v, u) is an arc (Ties are not permitted).
-- A tournament T is **transitive** if whenever (u, v) and (v, w) are arcs of T, then (u, w) is also an arc of T
-- As with graphs, a path P in a digraph D is a **Hamiltonian path** o f D if P contains all vertices of D
-- A cycle C in D is a **Hamiltonian cycle** if C contains every vertex of D
-- If D has a Hamiltonian cycle, then D is a **Hamiltonian digraph**

Chapter 8
Matchings and Factorization

-- A set of edges in a graph is **independent** if no two edges in the set are adjacent
-- By a **matching** in a graph G, we mean an independent set of edges i n G
-- We say that M **matches** the set { u 1, u 2, …, u k } to the set { w 1, w 2, …, w k
-- Let G be a bipartite graph with partite sets U and W such that | U | ≤ | W |. For a nonempty set X of U, the **neighborhood** N (X) of X is the union of the neighborhoods N (x), where x is in X
-- The graph G is said to satisfy **Hall' s condition** if | N (X)| ≥ | X | for every nonempty subset X of U
-- Then this collection of sets has a **system of distinct representatives** if there exist nondistinct elements x 1, x 2, …, x n such that x i S i for 1 ≤ i ≤ n

-- Such a matching is called a **maximum matching**
-- If a graph G of order 2k has a matching M of cardinality k, then this (necessarily maximum) matching M is called a **perfect matching** as M matches every vertex of G to some vertex of G
-- The **edge independence number** $\alpha'(G)$ of a graph G is the maximum cardinality of an independent set of edges
-- A vertex and an incident edge are said to **cover** each other
-- An **edge cover** of a graph G without isolated vertices is a set of edges of G that covers all vertices of G
-- The **edge covering number** $\beta'(G)$ of a graph G is the minimum cardinality of an edge cover of G
-- An edge cover of G of cardinality $\alpha'(G)$ is a **minimum edge cover** of G
-- A set of vertices in a graph is **independent** if no two vertices in the set are adjacent
-- The **vertex independence number** (or the **independence number)** (G) of a graph G is the maximum cardinality of an independent set of vertices in G
-- An independent set in G of cardinality (G) is called a **maximum independent set**
-- A **vertex cover** in a graph G is a set of vertices that covers all edges of G
-- The minimum number of vertices in a vertex cover of G is the **vertex covering number** (G) of G
-- A vertex cover of cardinality $\alpha(G)$ is a **minimum vertex cover** in G
-- A 1-regular spanning subgraph of a graph G is also called a **1- factor** of G
-- A component of a graph is **odd** or **even** according to whether its order is odd or even
-- A graph G is said to be **1-factorable** if there exist 1-factors F 1, F 2, …, F r of G such that { E (F 1), E (F 2), …, E (F r)} is a partition of E (G
-- We then say that G is **factored** into the 1-factors F 1, F 2, …, F r, which form a **1-factorization** of G
-- For this reason, the 1-factorization described in the proof is called a **cyclic factorization**
-- A **2-factor** in a graph G is a spanning 2-regular subgraph of G
-- A graph G is said to be **2-factorable** if there exist 2-factors F 1, F 2, …, F k such that { E (F 1), E (F 2), …, E (F k)} is a partition of E (G
-- A **Hamiltonian factorization** of a graph G is a 2-factorization of G in which each 2-factor is a Hamiltonian cycle
-- A graph G is **Hamiltonian-factorable** if there exists a Hamiltonian factorization of G

-- Hamiltonian factorization of K 9 More generally, a spanning subgraph F of a graph G is called a **factor** of G
-- The graph G is said to be **factorable** into the factors F 1, F 2, …, F k if {E (F 1), E (F 2), …, E (F k)} is a partition of E (G)
-- If each factor F i is isomorphic to some graph F, then G is **F-factorable**
-- A graph G is said to be **decomposable** into the subgraphs H 1, H 2, …, H k if { E (H 1), E (H 2), …, E (H k)} is a partition of E (G)
-- Such a partition produces a **decomposition** of G
-- If each H i is isomorphic to some graph H, then the graph G is **H-decomposable** and the decomposition is an **H-decomposable**
-- A **Steiner triple system** of order n is a set S of cardinality n and a collection T of 3-element subsets, called **triples,** such that every two distinct elements of S belong to a unique triple in T
-- From the K 3 - decomposition of K 7, we have now produced a Steiner triple system of order 7 from the set { 1, 2, …, 7 }, namely : The K 3 -decomposition is a **cyclic decomposition** of K 7
-- A one-to-one function f : V (G) → { 0, 1, 2, …, m } is called a **graceful labeling** of G if the induced edge labeling f ': E (G) → { 1, 2, …, m } defined by is also one-t o-one
-- The length of a smallest cycle in a graph is referred to as its **girth**
-- For an integer g ≥ 3, a **g-cage** is a 3-regular graph of minimum order that has girth g

Chapter 9
Planarity

-- A graph G is called a **planar graph** if G can be drawn in the plane so that no two of its edges cross each other
-- A graph that is not planar is called **nonplanar**
-- A graph G is called a **plane graph** if it is drawn in the plane so that no two edges of G cross
-- This problem is referred to as the **Three Houses and Three Utilities Problem**
-- A plane graph divides the plane into connected pieces called **regions**
-- This is the **exterior region**
-- The subgraph of a plane graph whose vertices and edges are Figure 9.5 : A planar graph and a plane graph incident with a given region R is the **boundary** of R
-- This is referred to as the **Euler Identity**

-- A graph G is **maximal planar** if G is planar but the addition of an edge between any two nonadjacent vertices of G results in a nonplanar graph
-- More formally, a graph G ' is called a **subdivision** of a graph G if G ' = G or one or more vertices of degree 2 are inserted into one or more edges of G
-- A " drawing " in plane is also called an **embedding** of G in the plane. In addition, we say that G can be **embedded** in the plane
-- A common surface is the **torus,** a doughnut-shaped surface (see Figure 9.19 (a)). In Figure 9.19 (b), we see that the graph K 4 can be embedded on the torus
-- The surface S k is also called a **surface of genus** k
-- What we have just observed then is that every graph can be **embedded** on some surface
-- The smallest nonnegative integer k such that a graph G can be embedded on S k is called the **genus** of G and is denoted by γ (G
-- A region is called a **2-cell** if any closed curve that is drawn in that region can be con tinuously contracted (or shrunk) in that region to a single point
-- An embedding, every region of which is a 2-cell, is called a **2-cell embedding**

Chapter 10
Coloring

-- The conjecture that every map can be colored with four or fewer colors became known as the **Four Color Conjecture** (proof based on a set that was later referred to as an **unavoidable set of reducible configurations).**
-- A **reducible configuration** is any arrangement of regions that cannot occur in a
-- By a **proper coloring** (or, more simply, a **coloring)** of a graph G, we mean an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently
-- The smallest number of colors in any coloring of a graph G is called the **chromatic number** of G and is denoted by λ(G). (The symbol is the Greek letter "chi .") If it is possible to color (the vertices of) G from a set of k colors, then G is said to be **k-colorable**
-- A coloring that uses k colors is called a **k-coloring**
-- If (G) = k, then G is said to be **k-chromatic** and every k-coloring of G is a **minimum coloring** of G
-- A **clique** in a graph G is a complete subgraph of G
-- The order of the largest clique in a graph G is its **clique number,** which is denoted by ω(G)
-- A **coloring** of a graph G can also be thought of as a function c from V (G) to the set **N** of positive integers (or natural numbers) such that adjacent vertices have

distinct functional values, that is, a **coloring** of G is a function c : V (G) → **N** such that uv E (G) implies that c (u) ≠ c (v)
-- This graph is **triangle-free** (it has no triangles) but has chromatic number 4
-- A graph G is called **perfect** if (H) = ω (H) for every induced subgraph H of G
-- An **edge coloring** of a nonempty graph G is an assignment of colors to the edges of G, one color to each edge, such that adjacent edges are assigned different colors
-- The minimum number of colors that can be used to color the edges of G is called the **chromatic index** (or sometimes the **edge chromatic number)** and is denoted by '(G)
-- An edge coloring that uses k colors is a **k-edge coloring**

Chapter 12
Distance

-- For two vertices u and v in a graph G, the **distance** d (u, v) from u to v is the length of a shortest u – v path in G
-- A u – v path of length d (u, v) is called a u – v **geodesic**
-- d (u, v) = d (v, u) for all u, v V (G) [**the symmetric property**]. 4
-- d (u, w) ≤ d (u, v)+ d (v, w) for all u, v, w V (G) [**the triangle inequality**]. That a connected graph satisfies all four of these properties should be clear, with the possible exception of property 4 (the triangle inequality), which we now verify
-- The fact that the distance d satisfies properties 1 – 4 means that d is a **metric** and (V (G), d) is a **metric space**
-- For a vertex v in a connected graph G, the **eccentricity** e (v) of v is the distance between v and a vertex farthest from v in G
-- The minimum eccentricity among the vertices of G is its **radius** and the maximum eccentricity is its **diameter,** which are denoted by rad (G) and diam (G), respectively
-- A vertex v in G is a **central vertex** if e (v) = rad (G) and the subgraph induced by the central vertices of G is the **center** Cen (G) of G
-- If every vertex of G is a central vertex, then Cen (G) = G and G is called **self-centered**
-- A vertex v in a connected graph G is called a **peripheral vertex** if e(v)=diam(G)
-- The subgraph of G induced by its peripheral vertices is the **periphery** Per (G) of G
-- Such a vertex v is called an **eccentric vertex of** u

-- A vertex v is an **eccentric vertex of the graph** G if v is an eccentric vertex of some ve
rtex of G

Chapter 13
Domination

-- The neighborhood (or **open neighborhood)** N(v) of v is the set of neighbors of v
-- The **closed neighborhood** N [v] is defined as N [v] = N (v) union {v}
-- A vertex v in a graph G is said to **dominate** itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood N [v]. Therefore, v dominates 1+deg v vertices of G
-- A set S of vertices of G is a **dominating set** of G if every vertex o f G is dominated by some vertex in S
-- G A **minimum dominating set** in a graph G is a dominating set of minimum cardinality
-- The cardinality of a minimum dominating set in G is called the **domination number** of G and is denoted by γ (G
-- If S is a dominating set of a graph G and no proper subset of S is a dominating set of G, then S is called a **minimal dominating set**. (In this context, a vertex does not dominate itself.) This type of domination is called **total domination**
-- A set S of vertices in a graph G is a **total dominating set** of G if every vertex of G is adjacent to at least one vertex of S
-- The minimum cardinality of a total dominating set is the **total domination number** $\gamma_t(G)$ of G
-- A total dominating set of cardinality $\gamma_t(G)$ is a **minimum total dominating set** for G